Cooperation in Wireless Ad Hoc Networks: A Market-Based Approach

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***Abstract*— We consider a market-based approach to stimulate cooperation in ad hoc networks where nodes charge a price for relaying data packets. Assuming that nodes set prices to maximize their own net benefit, we characterize the equilibria of the resulting market. In addition, we propose an iterative algorithm for the nodes to adapt their price and rate allocation, and study its convergence behavior. We use a numerical case study to illustrate our results.**

I. Introduction

A wireless ad hoc network is a collection of nodes which form a network independently of any fixed infrastructure. As opposed to networks which use routers to support network functions such as packet routing and forwarding, in ad hoc networks these functions are provided by the nodes (hosts) themselves. Initially, wireless ad hoc networks were studied in the realm of military or disaster relief situations; more recently, wireless ad hoc networks have also been envisioned for commercial applications such as providing Internet connectivity for nodes that are not in transmission range of a wireless access point.

The wireless medium and the infrastructureless nature of wireless ad hoc networks pose a variety of problems that are distinctly different from traditional networks. For example, routing algorithms for wireless ad hoc networks have to be able to cope with frequent and unpredictable topology changes. Nodes are generally battery powered; thus energy is a precious resource that has to be carefully managed by the nodes in order to avoid an early termination of their activity. Due to interference, ad hoc networks are generally characterized by bandwidth-constrained and variable-capacity links which makes bandwidth allocation difficult. The nature of ad hoc networks also challenges the traditional approach of separating network protocols into distinct layers in order to help handling the enormous complexity of network design. Due to energy constraints and the nature of the wireless channel, the interdependencies between layers in wireless ad hoc networks are so prominent that is often desirable to jointly design and optimize protocols at different layers. However, the above problems are challenging and currently not fully understood in the simpler case where layers are kept separate: as a result, the majority of the work in wireless ad hoc networks has focused on "single layer" protocol design and analysis. The motivation behind this approach is that a better understanding of each of these problems in isolation is essential before attempting cross-layer design and optimization.

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In addition to the characteristics outlined above, an important feature of wireless ad hoc networks is that nodes have to cooperate in relaying data packets for other nodes. In applications of wireless ad hoc networks for emergency and military situations all of the nodes belong to a single authority and cooperation among nodes can be assumed. However, in commercial applications nodes may not belong to a single authority but may in be fact independent entities that can freely decide on how they use their resources such as battery energy and transmission bandwidth. Since relaying a packet will incur a cost (of transmission bandwidth and energy) to a node, nodes may not necessarily volunteer to relay packets for other nodes. In this situation, an additional mechanism is needed to give nodes an incentive to cooperate. Several approaches to stimulate cooperation in wireless ad hoc networks have been proposed in the literature [3], [4], [6], [9], [10], [12], [14]. Roughly, these approaches can be classified into reputation-based and market-based (or credit-based) systems.

In reputation-based systems, nodes observe the behavior of other nodes and act accordingly by rewarding cooperative behavior and punishing uncooperative behavior. For example, Marti *et al.* proposed in [9] a reputation-based system where nodes use a "watchdog" mechanism to detect whether other nodes do relay data packets and avoid malicious nodes in their route selection. Michardi and Molva [10] proposed a system which provides three different reputation measures: subjective reputation (observations), indirect reputation (positive reports by others) and functional reputation (task-specific reputation). A weighted combination of these three reputation values is then used to make decisions about cooperation and to avoid/isolate malicious nodes. Buchegger and Le Boudec introduce in [4] a reputation-based system where nodes monitor the transmission activities of their neighbors to make sure that they relay data packets. If a neighbor does not forward packets, it is considered as uncooperative and this reputation is propagated through the network. All of the above reputation-based were evaluated purely through numerical case studies. Analytical approaches to study reputation-based systems include the following: Urpi *et al.* proposed in [16] a formal model based on Bayesian games to study reputation-based mechanisms such as the ones in [4], [10]. Using this approach, Urpi *et al.* derived a simple strategy that provably enforces packet relaying among nodes. Srivisin *et al.* proposed in [14] a reputation-based mechanism that takes energy into account, and provided a formal analysis using a game-theoretic framework based on the "tit-for-tat" strategy.

Another approach to creating incentives for packet relaying is the market-based (or credit-based) approach. In such a system, nodes receive a micro-payment (or credit) for every packet that they relay; in return, nodes can use these payments (credits) to send their own traffic. Buttyan and Hubaux proposed in [3], [5] a credit-based system where a node receives one unit of credit for relaying a message which are deducted from the sender. Their approach relies on a tamper-proof hardware at each node to ensure that the correct amount of credits are added or deducted. In [18], Zhong *et al.* consider a similar approach as in [3], [5]; however rather than requiring a tamper-free hardware, a secure protocol is used to manage the exchange of credits. A common characteristic of the above two approaches is that the price for relaying a packet is the same at each node in the network. To account for nodes having different amounts of resources available, two approaches that allow each node to set its own price have been presented in [6], [12].

In this paper, we consider a market-based approach. In order to focus on node cooperation with respect to packet forwarding (which is assumed to be implemented in the application layer), we adopt a "single-layer" approach and assume that routing and bandwidth allocation decision are done in lower layers (such as the transport and network layer). In particular, we assume that each node (a) sends traffic along a fixed route which is determined by a lower-layer routing mechanism and (b) is allocated a certain amount of bandwidth by a lower-layer rate control mechanism. Each node can then decide on

(a) the portion of the allocated bandwidth used to send its own traffic,
(b) the portion of the allocated bandwidth used to relay traffic for other nodes,
(c) the price it charges other nodes for relaying their traffic.

Nodes are assumed to be independent entities which act selfishly and choose a bandwidth allocation and a relay price in order to maximize their own net benefit. For this market, we wish to determine (a) whether there exists an equilibrium, (b) the degree of cooperation under an equilibrium, and (c) whether a market equilibrium is eventually reached. To simplify the discussion we will at first ignore battery constraints. Once we have established the results for this case, we will then show how the analysis can be extended to incorporate limited battery resources into the nodes allocation decision. We note that the exchange of micro-payments that are needed in our approach could be facilitated by secure protocols as proposed in [18].

The main differences between this paper and the other market-based approaches discussed above are as follows. Compared with [5], [18], we allow each node to freely decide on how much it charges for relaying traffic, whereas in [5], [18] it is assumed that the price for relaying is equal to one unit price per packet and is the same at all nodes. In addition, we do not impose any budget constraints on the nodes, *i.e.* we do not assume that nodes have to earn currencies (credits) by relaying traffic. Note that this condition would penalize nodes at the edge of the network where the traffic demand is low: these nodes may get little opportunity of relay packets, and hence may soon run out of currencies and become unable to access the network at all. Similar to our setting, the approaches proposed in [6], [12] allow nodes to charge different prices for relaying traffic; however, no formal justification for the rule mandated for setting the node prices is given. In particular, it is not verified whether it is in the best interest of each node to adopt the price adaptation rule proposed in [6], [12] and therefore a selfish node would indeed adopt these rules. The main contributions of this paper is to fill this gap by providing a formal analysis of the situation where each node can freely decide on how to allocate its transmission rate and on how much it charges other nodes for relaying their traffic.

The use of pricing as a means for allocating resources in communication networks has received much attention in recent years. In particular, the work by Kelly *et al.* ([7]) for wired networks spurred a plethora of research papers. In their work, Kelly *et al.* propose a scheme where a network provider charges nodes according to the traffic load on individual links in the network, and nodes accessing the network decide on their transmission rate based on these network prices. Kelly *et al.* show that this pricing scheme can be used to achieve (in equilibrium) a weighted proportional fair rate allocation. The market-based mechanism that we consider in this paper can be interpreted as a generalization of the pricing scheme of Kelly *et al.* to ad-hoc networks where each node generates its own traffic (acts as an end host) and relays traffic for other nodes (acts as a router). Without making this generalization, the framework of Kelly *et al.* has been directly applied to allocate bandwidth in wireless ad hoc networks [17] and stimulate cooperation among nodes in a wireless network [6].

The rest of the paper is organized as follows. In Section II, we define more precisely the market-based bandwidth mechanism that we consider. In Section III, we study the existence of a market equilibrium. In Section IV we derive the iterative algorithm and study in Section V its convergence behavior. In Section VII we provide a numerical case study to illustrate our results. In Section VIII, we discuss how our model can be extended to incorporate battery constraints.

II. Problem Formulation

Consider an ad hoc network which consists of a set of nodes $\mathcal{N} = \{1, 2, \cdots, N\}$. Each node sends traffic to a single destination along a fixed route r_n, where $r_n \subset \mathcal{N}$ is the set of nodes that relay traffic for node n. Let $G(n)$ be set of nodes which send (relay) traffic through node n and let $H(n) = G(n) \cup \{n\}$. The routing matrix $A = (A_{nm}; n, m \in \mathcal{N})$ is then given by

$$A_{nm} = \begin{cases} 1 & \text{if } m \in H(n), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A_{nn} = 1, n \in \mathcal{N}$.

Let C_n be the total transmission rate that is allocated to node n by lower layer protocols. Each node $n \in \mathcal{N}$ can freely decide on (a) how it allocates the bandwidth C_n to send its own traffic and relay traffic for other nodes, and (b) how much it charges other nodes for relaying their traffic. Let (x_n, y_n, μ_n) denote the allocation at node n where x_n is the rate at which node n transmits its own traffic, y_n is the rate allocated for relaying traffic for other nodes, and μ_n be the price (per unit flow) that node n charges other nodes for relaying their traffic. Naturally, we have the capacity constraint that $x_n + y_n \leq C_n$. In the following, we model how node n decides on its allocation (x_n, y_n, μ_n).

A. Node Utility Function

We use the function $U_n(x_n)$ to characterize the utility that node $n \in \mathcal{N}$ obtains by sending its own traffic at rate x_n.

Assumption 1: The utility function $U_n : [0, C_n] \to \Re_+$ is increasing, strictly concave, and we have $U_n(0) = 0$. Furthermore, U_n is continuously differentiable and its first derivative U_n' such that $U_n'(C_n) = 0$.

Traffic that is characterized by a utility function as given by Assumption 1 is referred to as *elastic traffic* [11]. We note that utility functions with these characteristics are commonly used in the pricing literature (see for example [7]). Throughout the paper, we assume that the utility function U_n is private information known to node n, but not to other nodes.

Using the utility function U_n, we define the demand function $D_n(\lambda_n)$ by

$$D_n(\lambda_n) = \arg \max_{0 \le x_n \le C_n} \{U_n(x_n) - \lambda_n x_n\}, \quad \lambda_n \ge 0.$$

The value $D_n(\lambda_n)$ is equal to the rate x_n at which node n sends its own traffic when the cost per unit transmission rate is equal to λ_n. Using Assumption 1, the demand function D_n is given by

$$D_n(\lambda_n) = \begin{cases} U_n'^{-1}(\lambda_n) & 0 \le \lambda_n \le U_n'(0), \\ 0 & \text{otherwise}, \end{cases}$$

and we have $D_n(0) = C_n$. Note that the demand function D_n is non-increasing. We impose an additional assumption on the utility function U_n to ensure the the demand function satisfies a Lipschitz condition. We will use this technical assumption in our analysis in Section IV and V.

Assumption 2: The utility function U_n is such that the corresponding demand function D_n satisfies the condition

$$|D_n(\lambda) - D_n(\lambda')| \le L_n |\lambda - \lambda'|, \qquad \lambda, \lambda' \ge 0,$$

for some constant L_n.

We note that Assumption 1 and 2 are standard assumption used in the literature on market-based (or utility-based) bandwidth allocation [8].

B. Relay Traffic

Besides sending its own traffic, node n can also forward traffic for other nodes. We call the total amount of traffic that node n can potentially forward for other nodes the *relay traffic* at node n. Using the discussion of the previous subsection, the relay traffic at node n will be a function of the price μ_n at node n: increasing μ_n will tend to decrease the relay traffic at node n, and vice versa. However, the relay traffic also depends on prices at other nodes. For example, when node n charges a low price μ_n, but all other nodes charge a high price for their relay service, then the relay traffic at node n might still be low. Accordingly, we characterize the relay traffic at node n by a function $I_n(\mu_n, \mu_{-n})$ where μ_n is the price charged by node n and $\mu_{-n} \in \Re_+^{N-1}$ is the vector indicating the prices set by nodes other than n. Due to the coupling between price and traffic loss, deriving the exact expression of function $I_n(\mu_n, \mu_{-n})$ would require to obtain the closed-form solution for a fixed-point of a non-linear system of equation. It is not clear whether this can be done; indeed the main challenge in studying the above market-based approach is the characterization of the relay traffic. Rather than trying to obtain the exact expression, we characterize the basic properties of the function I_n that we use in our analysis. In particular, we make the following assumptions.

Assumption 3: The function $I_n(\mu_n, \mu_{-n})$, $(\mu_n, \mu_{-n}) \in \Re_+^N$, has the following properties.

(a) I_n is continuous and non-increasing in μ_n.
(b) For every fixed $\mu_{-n} \in \Re_+^{N-1}$, I_n is continuously differentiable with respect to μ_n except at a finite subset A of $\Re_+$.
(c) If $I_n(\mu_n, \mu_{-n})$ is not differentiable with respect to μ_n at $(\bar{\mu}_n, \bar{\mu}_{-n}) \in \Re_+^N$, then there exists a neighborhood B around $(\bar{\mu}_n, \bar{\mu}_{-n})$ and a continuous function $g_n : \Re_+^{N-1} \mapsto \Re_+$ such that $I_n(\mu_n, \mu_{-n})$ is not differentiable with respect to μ_n at a point $(\mu_n', \mu_{-n}') \in B$ if and only if $\mu_n' = g_n(\mu_{-n}')$.
(d) There exists a constant H_n such that for all $\mu_n, \mu_n' \in \Re_+$ and $\mu_{-n} \in \Re_+^{N-1}$ we have

$$|I_n(\mu_n, \mu_{-n}) - I_n(\mu_n', \mu_{-n})| \le H_n |\mu_n - \mu_n'|.$$

Let us briefly comment on the above assumption. The assumption given by property (a) that the relay traffic is continuous and non-increasing in μ_n is intuitive. Property (b) states that for a fixed $\mu_{-n} \in \Re_+^{N-1}$ there exists at most a finite number of points where the function $I_n(\mu_n, \mu_{-n})$ is not differentiable with respect to μ_n. The motivation for this property is as follows. Consider a fixed price vector μ_{-n} and suppose there exists an upstream node n' of node n such that under the price $\bar{\mu}_n$ node n' is not congested, but any decrease of the price at node n below $\bar{\mu}_n$ causes traffic loss at the upstream node n'. In this case, the function $I_n(\mu_n, \mu_{-n})$ might not be differentiable at $\bar{\mu}_n$ as the traffic loss at the upstream now may lead to a discontinuity in the slope of the relay traffic. As there are a finite number of links, the number of such discontinuities is finite. Property (c) states that the discontinuities in the derivative of I_n with respect to μ_n are given by continues functions. Under the assumption that the function I_n is continuous as given by property (b) and discontinuities occur when links become congested, then the discontinuities in the derivative of I_n with respect to μ_n can be characterized by continuous functions as given by property (c). The Lipschitz condition given by property (d) is used in the convergence analysis of Section V.

The fact that the function I_n is not everywhere differentiable with respect to μ_n will complicate the analysis, and properties (b) and (c) of Assumption 3 will play an important role in establishing our results. In particular, property (b) implies that the right derivative

$$I_{r,n}'(\mu_n, \mu_{-n}) = \lim_{\delta \downarrow 0} \frac{I_n(\mu_n + \delta, \mu_{-n}) - I_n(\mu_n, \mu_{-n})}{\delta}$$

and the left derivative

$$I_{l,n}'(\mu_n, \mu_{-n}) = \lim_{\delta \uparrow 0} \frac{I_n(\mu_n + \delta, \mu_{-n}) - I_n(\mu_n, \mu_{-n})}{\delta}$$

exist for all $(\mu_n, \mu_{-n}) \in \Re_+^N$, where we use the convention that

$$I_{l,n}'(0, \mu_{-n}) \triangleq I_{r,n}'(0, \mu_{-n}), \qquad \mu_{-n} \in \Re_+^{N-1}.$$

Note that without having the above convention, the left derivative is not defined for $\mu_n = 0$.

C. Node Problem

Consider a given node $n \in \mathcal{N}$ and suppose that the prices charged at nodes other than n are fixed and given by the vector $\mu_{-n} \in \Re_+^{N-1}$. Furthermore, let

$$\lambda_n = \sum_{m \in r_n} \mu_m$$

be the price (per unit traffic) that node n has to pay for sending its own traffic under the price vector μ_{-n}. The net benefit (total payoff) that node n obtains under an allocation (x_n, y_n, μ_n) is then given by

$$U_n(x_n) - x_n\lambda_n + \mu_n \min\{y_n, I_n(\mu_n, \mu_{-n})\},$$

where $U_n(x_n)$ is the utility of node n under the bandwidth allocation x_n, the term $x_n\lambda_n$ reflects the total cost that node n has to pay for sending its traffic along route r_n, and the term $(\mu_n \min\{y_n, I_n(\mu_n, \mu_{-n})\})$ represents the revenue of node n made by relaying traffic for other nodes. Note that when the relay traffic $I_n(\mu_n, \mu_{-n})$ is less than the bandwidth y_n that node n has allocated for forwarding traffic for other nodes, then the total revenue that node n makes by relaying traffic for other nodes is given by $\mu_n I_n(\mu_n, \mu_{-n})$.

To maximize its net benefit, node n will choose an allocation (x_n, y_n, μ_n) which solves the node problem $NODE(U_n, C_n, I_n, \mu_{-n})$ given by

$$\begin{aligned} \max_{x_n, y_n, \mu_n} \quad & U_n(x_n) - x_n\lambda_n + y_n\mu_n, \\ \text{subject to} \quad & x_n + y_n \leq C_n, \\ & y_n \leq I_n(\mu_n, \mu_{-n}), \\ & x_n, y_n, \mu_n \geq 0. \end{aligned} \tag{1}$$

We characterize the optimal solutions to the above optimization problem in Section III-A. For now, we note that when the optimal solution (x_n^*, y_n^*, μ_n^*) to the node problem given by Eq. (1) is such that $x_n^* + y_n^* < C_n$, then the price μ_n^* is given by

$$\mu_n^* = \arg\max_{\mu_n \geq 0} \mu_n I_n(\mu_n, \mu_{-n}). \tag{2}$$

The following assumption ensures that the above optimization problem has a unique solution. Let $\kappa_{r,n}(\mu_n, \mu_{-n})$ be given by

$$\kappa_{r,n}(\mu_n, \mu_{-n}) = -\frac{I_n(\mu_n, \mu_{-n})}{I'_{r,n}(\mu_n, \mu_{-n})}, \quad (\mu_n, \mu_{-n}) \in \Re_+^N,$$

where we use the convention that $\kappa_{r,n}(\mu_n, \mu_{-n}) = 0$ if we have that $I_n(\mu_n, \mu_{-n}) = I'_{r,n}(\mu_n, \mu_{-n}) = 0$. Similarly, let $\kappa_{l,n}(\mu_n, \mu_{-n})$ be given by

$$\kappa_{l,n}(\mu_n, \mu_{-n}) = -\frac{I_n(\mu_n, \mu_{-n})}{I'_{l,n}(\mu_n, \mu_{-n})}, \quad (\mu_n, \mu_{-n}) \in \Re_+^N,$$

where we use the convention that $\kappa_{l,n}(\mu_n, \mu_{-n}) = 0$ for the case where $I_n(\mu_n, \mu_{-n}) = I'_{l,n}(\mu_n, \mu_{-n}) = 0$.

If $I_n(\mu_n, \mu_{-n})$ is differentiable with respect to μ_n at (μ_n, μ_{-n}), then we define

$$\kappa_n(\mu_n, \mu_{-n}) = -\frac{I_n(\mu_n, \mu_{-n})}{I'_n(\mu_n, \mu_{-n})}.$$

Assumption 4: The functions $\kappa_{r,n}(\mu_n, \mu_{-n})$ and $\kappa_{l,n}(\mu_n, \mu_{-n})$ are bounded and non-increasing in μ_n.

The above assumption implies that

$$\kappa_{l,n}(\mu_n, \mu_{-n}) \geq \kappa_{r,n}(\mu_n, \mu_{-n}), \qquad (\mu_n, \mu_{-n}) \in \Re_+^N.$$

Lemma 1: Under Assumption 4, if $I_n(0, \mu_{-n}) > 0$ then the optimal solution to the optimization problem given by Eq. (2) is equal to the unique price μ_n^*, $\mu_n^* \geq 0$, such that $\kappa_{l,n}(\mu_n^*, \mu_{-n}) \geq \mu_n^* \geq \kappa_{r,n}(\mu_n^*, \mu_{-n})$.

Proof: By Assumption 4, the function $\kappa_{r,n}(\mu_n, \mu_{-n})$ is non-negative, bounded and non-increasing in μ_n. Furthermore, $\kappa_{r,n}(0, \mu_{-n})$ is positive when $I_n(0, \mu_{-n}) > 0$. Combining these facts, it follows that there exists a unique price $\mu_n^* \geq 0$ such that $\mu_n < \kappa_{r,n}(\mu_n, \mu_{-n})$ for all $\mu_n < \mu_n^*$, and $\mu_n > \kappa_{r,n}(\mu_n, \mu_{-n})$ for all $\mu_n > \mu_n^*$. Using Assumption 4, this implies that $\kappa_{l,n}(\mu_n^*, \mu_{-n}) \geq \mu_n^* \geq \kappa_{r,n}(\mu_n^*, \mu_{-n})$, and for every price μ_n that is different from μ_n^* we either have that $\mu_n < \kappa_{r,n}(\mu_n, \mu_{-n})$ or $\mu_n > \kappa_{l,n}(\mu_n, \mu_{-n})$. It is straightforward to verify that μ_n can not be an optimal solution if we have that $\mu_n < \kappa_{r,n}(\mu_n, \mu_{-n})$ or $\mu_n > \kappa_{l,n}(\mu_n, \mu_{-n})$, as in this case we can increase the value of the objective function in Eq. (1) by slightly increasing μ_n, or decreasing μ_n, respectively. Hence, the price μ_n^* is the unique optimal solution to the optimization problem given by Eq. (2). ∎

For a given price vector $\mu_{-n} \in \Re_+^{N-1}$, let $\beta_n(\mu_{-n})$ be the optimal solution to the maximization problem given by Eq. (2), where we use the convention that $\beta_n(\mu_{-n}) = 0$ if $I_n(\mu_n, \mu_{-n}) = 0$ for all $\mu_n \in \Re_+$.

The functions $\kappa_{r,n}$ and $\kappa_{l,n}$ characterize the elasticity of the relay traffic I_n, where we say that the relay traffic is elastic (inelastic) if a small (large) change in the price μ_n leads to a large (small) change in the relay traffic I_n. The relay traffic is elastic (inelastic) at (μ_n, μ_{-n}) if $\kappa_{r,n}(\mu_n, \mu_{-n})$ and $\kappa_{l,n}(\mu_n, \mu_{-n})$ are small (large). Lemma 1 implies that the optimal solution to the maximization problem given by Eq. (2) is small (large) if the relay traffic at node n is elastic (inelastic).

III. Market Equilibrium

Having characterized the allocation decision at an individual node, we consider in this section the situation where all nodes simultaneously try to solve the node problem given by Eq. (1), and study the existence and properties of a market equilibrium. We use the following notation. Let $e = (x_n, y_n, \mu_n)$ be the allocation at node n and let $e = (e_1, ..., e_N)$ be the vector indicating the allocation decisions at all nodes $n \in \mathcal{N}$.

Definition 1: We call a vector $e^* = (e_1^*, ..., e_N^*)$ an equilibrium allocation if for every node $n \in \mathcal{N}$ we have that e_n^* is an optimal solution to $NODE(U_n, C_n, I_n, \mu_{-n}^*)$.

Note that under an equilibrium allocation e^* each node maximizes its own net benefit and therefore has no incentive to deviate from its allocation e_n^*. We refer to an equilibrium allocation e^* as a *market equilibrium.*

In Section III-A, we characterize the optimal solution for the node problem given by Eq. (1). We then use this result in Section III-B to show that there always exists an equilibrium allocation and characterize the bandwidth allocation under a market equilibrium.

A. Optimal Solution to Node Problem

In this section, we characterize the optimal solutions for the node problem $NODE(U_n, C_n, I_n, \mu_{-n})$.

Lemma 2: Under Assumptions 1, 3, and 4, if the allocation (x_n^*, y_n^*, μ_n^*) is an optimal solution for the node problem $NODE(U_n, C_n, I_n, \mu_{-n})$ then we have that $y_n^* = I_n(\mu_n^*, \mu_{-n})$ and $\mu_n^* \geq \beta_n(\mu_{-n})$.

Note that the equality $y_n^* = I_n(\mu_n^*, \mu_{-n})$ implies that there is no traffic loss at node n under an optimal allocation (x_n^*, y_n^*, μ_n^*).

Proof: Note that the lemma trivially holds when $I_n(0, \mu_{-n}) = 0$, and we assume that $I_n(0, \mu_{-n}) > 0$. Suppose that $y_n^* < I_n(\mu_n^*, \mu_{-n})$. By assumption, the function $I_n(\mu_n^*, \mu_{-n})$ is continuous and decreasing. Therefore, there exists a price $\hat{\mu}_n$ such that $\hat{\mu}_n > \mu_n^*$ and $y_n^* < I_n(\hat{\mu}_n, \mu_{-n}) \leq I_n(\mu_n^*, \mu_{-n})$. Using these observations, we obtain for the allocation $(x_n^*, y_n^*, \hat{\mu}_n)$ that

$$\begin{aligned} &\Big(U_n(x_n^*) - x_n^*\lambda_n) + y_n^*\hat{\mu}_n\Big) - ... \\ &\quad -\Big(U_n(x_n^*) - x_n^*\lambda_n + y_n^*\mu_n^*\Big) \\ &= y_n^*(\hat{\mu}_n - \mu_n^*) > 0, \end{aligned}$$

which contradicts the fact that the allocation (x_n^*, y_n^*, μ_n^*) is an optimal solution; hence we have $y_n^* = I_n(\mu_n^*, \mu_{-n})$.

Similarly, for the case where $\mu_n^* < \beta_n(\mu_{-n})$ one can show that node n can improve its net benefit by increasing the price μ_n^*. ■

Using the above lemma, we obtain the following result.

Proposition 1: Under Assumptions 1, 3, and 4, there exists an optimal solution to $NODE(U_n, C_n, I_n, \mu_{-n})$. Furthermore, when $I_n(0, \mu_{-n})$ is positive, *i.e.* we have $I_n(0, \mu_{-n}) > 0$, then there exists a unique optimal solution and (x_n^*, y_n^*, μ_n^*) is an optimal allocation if and only if there exists κ_n, $\kappa_{r,n}(\mu_n^*, \mu_{-n}) \leq \kappa_n \leq \kappa_{l,n}(\mu_n^*, \mu_{-n})$, such that $\mu_n^* \geq \kappa_n$ and

$$\begin{aligned} x_n^* &= D_n\Big(\mu_n^* - \kappa_n + \lambda_n\Big) \\ y_n^* &= I_n(\mu_n^*, \mu_{-n}) \\ \mu_n^* &= \kappa_n, \qquad \text{if } x_n^* + y_n^* < C_n. \end{aligned}$$

When $I_n(0, \mu_{-n}) = 0$, then the allocation $x_n^* = D_n(\lambda_n)$ and $y_n^* = \mu_n^* = 0$ is an optimal solution.

We provide a proof for the above Proposition in Appendix A.

B. Existence of a Market Equilibrium

Using Proposition 1, we immediately obtain the necessary and sufficient conditions for an equilibrium allocation.

Proposition 2: Under Assumption 1 3, and 4, the allocation vector $e^* = (e_1^*, ..., e_N^*)$ is an equilibrium allocation if and only if for every node $n \in \mathcal{N}$ there exists κ_n, $\kappa_{r,n}(\mu_n^*, \mu_{-n}^*) \leq \kappa_n \leq \kappa_{l,n}(\mu_n^*, \mu_{-n}^*)$, such that $\mu_n^* \geq \kappa_n$ and

$$\begin{aligned} x_n^* &= D_n\Big(\mu_n^* - \kappa_n + \lambda_n^*\Big) \\ y_n^* &= I_n(\mu_n^*, \mu_{-n}^*) \\ \mu_n^* &= \kappa_n, \qquad \text{if } x_n^* + y_n^* < C_n. \end{aligned}$$

The following lemma characterizes the bandwidth allocation under an equilibrium.

Lemma 3: Let e^* be an equilibrium allocation and let κ_n, $n \in \mathcal{N}$, be the corresponding constants as given in Proposition 2. Under Assumption 1, 3, and 4, the bandwidth allocation $x^* = (x_n^*, n \in \mathcal{N})$ under e^* is the unique optimal solution to the maximization problem

$$\begin{aligned} &\max_{x \in \Re_+^N} \sum_{n \in \mathcal{N}} \left(U_n(x_n) - x_n \sum_{m \in r_n} \kappa_m \right) \qquad (3) \\ &\text{subject to} \qquad Ax \leq C. \end{aligned}$$

Proof: By Assumption 1, for each node n the utility function U_n is strictly concave on $[0, C_n]$ and the above optimization problem has a unique solution. Using the the Lagrange multiplier θ_n for the capacity constraint at node n, the optimization problem given by Eq. (3) can be rewritten as

$$\begin{aligned} &\max_{x \in \Re_+^N} \sum_{n \in \mathcal{N}} \left(U_n(x_n) - x_n \sum_{m \in r_n} \kappa_m \right) - ... \\ &\quad - \sum_{n \in \mathcal{N}} \theta_n \Big(\sum_{m \in H(n)} x_m - C_n \Big) \end{aligned}$$

where we set $U_n(x_n) = U_n(C_n)$ for $x_n \geq C_n$. The above maximization problem is of the same form as the system problem considered by Kelly *et al.* in [7]. Using the result from [7] we then obtain that the bandwidth allocation x^* under the Nash equilibrium e^* is an optimal solution, if there exists a vector $\theta^* = (\theta_1^*, ..., \theta_N^*) \in \Re_+^N$ such that

$$x_n^* = D_n \left(\theta_n^* + \sum_{m \in r_n} (\theta_m^* + \kappa_m) \right),$$

and $\theta_n^* = 0$ if $\sum_{m \in H(n)} x_m^* < C_n$. Using Proposition 2, we can construct such a vector by setting $\theta_n^* = \mu_n^* - \kappa_n$, $n \in \mathcal{N}$, where μ_n^* is the price at the Nash equilibrium e^* at node n. ■

As a special case, we consider the limiting case where $\kappa_{l,n}(\mu, \mu_{-n})$ approaches 0 for all nodes $n \in \mathcal{N}$ and all $(\mu, \mu_{-n}) \in \Re_+^N$. This corresponds to the case where the relay traffic at each node n is very elastic, *i.e.* a small change in the price μ_n will lead to a large change in the relay traffic I_n. Using Lemma 3, in this case the bandwidth allocation x^* at a Nash equilibrium solves the following optimization problem

$$\begin{aligned} &\max_{x \in \Re_+^N} \qquad \sum_{n \in \mathcal{N}} U_n(x_n) \\ &\text{subject to} \qquad Ax \leq C. \end{aligned}$$

This means that the bandwidth allocation under an equilibrium allocation maximizes the sum of the utilities over all users, *i.e.*

the bandwidth allocation is socially optimal bandwidth allocation. In the general case (given by Lemma 3), the bandwidth allocation under an equilibrium allocation will deviate from a socially optimal allocation; the more inelastic the relay traffic, the larger the deviation will be.

The next proposition states that there always exists an equilibrium allocation.

Proposition 3: Under Assumptions 1, 3, and 4, there always exists an equilibrium allocation e^*.
We provide a proof for the above Proposition in Appendix B.

IV. Node Adaptation

The analysis of the previous subsection is conceptually important as it allows to determine (a) whether an equilibrium exists for the market-based mechanism that we consider and (b) how the bandwidth is shared among nodes under an equilibrium allocation. However, it does not lend itself to derive an algorithm which allows each node n to compute an optimal solution to the node problem given by Eq. (1) based on information that is locally available. To see this, note that solving the node problem requires global knowledge of the prices charged at other nodes, as well as the utility functions of other nodes, to determine the relay traffic function $I_n(\mu_n, \mu_{-n})$. As it is not possible to solve the node problem directly, node n could try to use an iterative algorithm that increase its net benefit at each iteration step. Below we describe such an approach where node n uses the parameter $\hat{\kappa}_n$, $\hat{\kappa}_n \geq 0$, to estimate the value of the parameter κ_n associated with an optimal solution to the node problem as given by Proposition 1.

Suppose that node n updates its allocation at discrete iteration steps $k = 0, 1, ...$ Let μ_n^{k-1} be the price at node n at iteration step $k-1$ and let λ_n^{k-1} that node n has to pay for sending its own traffic at iteration step $k-1$. Furthermore, let i_n^{k-1} be the total amount of traffic by other nodes (relay traffic) that node n could relay at iteration $k-1$, and let $\hat{\kappa}_n$, $\kappa_n \geq 0$, be an estimate for the parameter κ_n associated with an optimal solution to the node problem as given by Proposition 1. Node n then chooses its allocation (x_n^k, y_n^k, μ_n^k) as follows. Keeping the price μ_n^{k-1}, node n chooses first a bandwidth allocation (x_n^k, y_n^k) which solves the following maximization problem,

$$\begin{array}{ll} \max\limits_{x_n, y_n} & U_n(x_n) - x_n\lambda_n^{k-1} + y_n(\mu_n^{k-1} - \hat{\kappa}_n) \\ \text{subject to} & x_n + y_n \leq C_n \\ & y_n \leq i_n^{k-1} \\ & x_n, y_n \geq 0. \end{array} \quad (4)$$

Note that this maximization problem is similar to the node problem given by Eq. (1), however, now the function $I_n(\mu_n, \mu_{-n}^{k-1})$ is replaced by the observed relay traffic i_n^{k-1} at time slot $k-1$. In addition, the constant $\hat{\kappa}_n$ is used as an estimate of the value κ_n that characterizes an optimal solution as given in Proposition 1 . After deciding on (x_n^k, y_n^k), node n updates its price by setting

$$\mu_n^k = \max\left\{\hat{\kappa}_n, \mu_n^{k-1} + \alpha_n d_n^k\right\}, \quad (5)$$

where $\alpha_n > 0$ is a small step size parameter and the update direction d_n^k is given by

$$d_n^k = D_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right) + i_n^{k-1} - C_n.$$

The above price update rule has roughly the following interpretation. When price μ_n^{k-1} is too low such that the total demand exceeds the transmission capacity C_n (*i.e., when* $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} > C_n$), then node n increases the price by a little bit to reduce demand. When price μ_n^{k-1} is too high and node n has spare capacity (*i.e., when* $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} < C_n$), then node n decreases the price by a little bit to stimulate demand; however, node n will never lower its price below $\hat{\kappa}_n$.

Below, we first characterize the optimal solution to the bandwidth allocation problem given by Eq. (4). We then use this result to show that the price update given by Eq. (5) indeed tends to increase at each iteration step the net benefit of node n. For the analysis, we assume that the estimate $\hat{\kappa}_n$ that node n uses to update its allocation (x_n^k, y_n^k, μ_n^k) is fixed and does not change from one iteration step to the next.

A. Bandwidth Allocation

When $x_n + y_n = C_n$, then we have $y_n = C_n - x_n$ and the value of the objective function given by Eq. (4) is given by

$$Q_{h,n}^k(x_n) = U_n(x_n) - x_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + C_n\mu_n^{k-1}.$$

Similarly, when $y_n = i_n^{k-1}$ then the value of the objective function given by Eq. (4) is given by

$$Q_{l,n}^k(x_n) = U_n(x_n) - x_n\lambda_n^{k-1} + i_n^{k-1}(\mu_n^{k-1} - \hat{\kappa}_n).$$

In the following we will use the functions $Q_{h,n}^k$ and $Q_{l,n}^k$ to characterize an optimal solution (x_n^k, y_n^k) to the maximization problem given by Eq. (4). We consider separately the cases where $x_n^k + i_n^{k-1} \geq C_n$, and $x_n^k + i_n^{k-1} < C_n$. Note that under Assumption 1, both functions $Q_{h,n}^k(x_n)$ and $Q_{h,n}^k(x_n)$ are strictly concave on $[0, C_n]$, and we have that

$$D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) = \arg \max_{x_n \in [0, C_n]} Q_{h,n}^k(x_n)$$

and

$$D_n(\lambda_n^{k-1}) = \arg \max_{x_n \in [0, C_n]} Q_{l,n}^k(x_n).$$

In addition, we have the following lemma.

Lemma 4: Under Assumption 1, the function $Q_{h,n}^k$ is increasing on the interval $[0, D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}))]$ and decreasing on $[D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1})), C_n]$. Similarly, the function $Q_{l,n}^k$ is increasing on the interval $[0, D_n(\lambda_n^{k-1})]$ and decreasing on $[D_n(\lambda_n^{k-1}), C_n]$.
The above lemma follows directly from the Assumption 1 and we omit a proof.

Let $\bar{x}_n^k$ be given by

$$\bar{x}_n^k = \max\{0, C_n - i_n^k\}.$$

Using this definition, let $x_{h,n}^k$ and $x_{l,n}^k$ be given by

$$x_{h,n}^k = \arg \max_{x_n \in [\bar{x}_n^k, C_n]} Q_{h,n}^k(x_n)$$

and

$$x_{l,n}^k = \arg \max_{x_n \in [0, \bar{x}_n^k]} Q_{l,n}^k(x_n).$$

Note that by Lemma 4 we have that $x_{h,n}^k$ and $x_{l,n}^k$ are well defined. Using the first order conditions for the optimization problem given by Eq. (4), we obtain the following results (we omit here a detailed derivation).

Lemma 5: Under Assumption 1, we have $Q_{h,n}^k(\bar{x}_n^k) \geq Q_{l,n}^k(\bar{x}_n^k)$ and the equation holds with equality when $\bar{x}_n^k > 0$.

The next lemma shows that we can use the functions $Q_{h,n}^k$ to characterize an optimal solution (x_n^k, y_n^k) to the maximization problem given by Eq. (4) for the case where $x_n^k + i_n^{k-1} \geq C_n$.

Lemma 6: Under Assumption 1, when (x_n^k, y_n^k) is an optimal solution to the maximization problem given by Eq (4) such that $x_n^k + i_n^{k-1} \geq C_n$, then we have that $x_n^k = x_{h,n}^k$ and $y_n^k = C_n - x_n^k$.

Similarly, the next lemma shows that we can use the functions $Q_{l,n}^k$ to characterize an optimal solution (x_n^k, y_n^k) to the maximization problem given by Eq. (4) for the case where $x_n^k + i_n^{k-1} < C_n$.

Lemma 7: Under Assumption 1, when (x_n^k, y_n^k) is an optimal solution to the maximization problem given by Eq (4) such that $x_n^k + i_n^{k-1} < C_n$, then we have that $x_n^k = x_{l,n}^k$ and $y_n^k = i_n^k$.

Using Lemma 6 and 7, it suffices to consider the function $Q_{h,n}^k(x_n)$ on $[\bar{x}_n^k, C_n]$ and $Q_{l,n}^k(x_n)$ on $[0, \bar{x}_n^k]$ to find the optimal allocation (x_n^k, y_n^k). In particular, we have that x_n^k is equal to $x_{h,n}^k$ if $Q_{h,n}^k(x_{h,n}^k)$ is larger or equal to $Q_{l,n}^k(x_{l,n}^k)$; otherwise x_n^k is equal to $x_{l,n}^k$. Using this observation, we obtain the following results.

Lemma 8: Under Assumption 1, when $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} \geq C_n$, then the optimal solution (x_n^k, y_n^k) for Eq. (4) is given by $x_n^k = D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1})$ and $y_n^k = C_n - x_n^k$.

Proof: The condition that

$$D_n(\mu_n^{k-1} - \kappa_n + \lambda_n^{k-1}) + i_n^{k-1} \geq C_n$$

implies that

$$\bar{x}_n^k \leq D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) \leq D_n(\mu_n^{k-1} + \lambda_n^{k-1})$$

and by Lemma 5 we have that $Q_{l,n}^k(x_n)$ is increasing on $[0, \bar{x}_n]$ and $x_{l,n}^k = \bar{x}_n^k$. Combining this result with the fact that by Lemma 5 we have

$$Q_{h_n}^k(\bar{x}_n^k) \geq Q l_n{}^k(\bar{x}_n^k),$$

we obtain that

$$Q_{h,n}^k(x_{h,n}^k) \geq Q_{h,n}^k(\bar{x}_n^k) \geq Q_{l_n}^k(\bar{x}_n^k) = Q_{l,n}^k(x_{l,n}^k).$$

The lemma then follows. ■

Lemma 9: Under Assumption 1, when we have that $D_n(\lambda_n^{k-1}) + i_n^{k-1} < C_n$, then the optimal solution (x_n^k, y_n^k) to (4) is given by $x_n^k = D_n(\lambda_n^{k-1})$ and $y_n^k = i_n^{k-1}$.

Lemma 10: Under Assumption 1, when $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} < C_n$ and $D_n(\lambda_n^{k-1}) + i_n^{k-1} \geq C_n$, then the optimal solution to the maximization problem (4) is given by $x_n^k = C_n - i_n^{k-1}$ and $y_n^k = i_n^{k-1}$.

Lemma 9 and 10 can be proved using the same line of argument as given for Lemma 8; we omit a detailed derivation. The above lemmas characterize the bandwidth allocation chosen by node n at iteration step k; the resulting bandwidth allocation algorithm is given in Fig. 1.

B. Price Update

Having established how node n should update its bandwidth allocation, we derive next price update rule given by Eq. (5). We use the following notation. Consider the function $R_n^k(\mu_n)$ given by

$$R_n^k(\mu_n) = U_n(\bar{x}_n^*) - \bar{x}_n^* \lambda_n^{k-1} + \bar{y}_n^* \mu_n, \qquad \mu_n \geq 0, \qquad (6)$$

where for the given value μ_n the rate allocation $(\bar{x}_n^*, \bar{y}_n^*)$ is the optimal solutions to the bandwidth allocation problem given by Eq. (4), *i.e.* $(\bar{x}_n^*, \bar{y}_n^*)$ is the optimal solutions to the maximization problem

$$\begin{aligned} &\max_{x_n, y_n} U_n(x_n) - x_n \lambda_n^{k-1} + y_n(\mu_n - \hat{\kappa}_n) \\ &\text{subject to} \qquad x_n + y_n \leq C_n \\ &\qquad\qquad\qquad y_n \leq i_n^{k-1} \\ &\qquad\qquad\qquad x_n, y_n \geq 0. \end{aligned}$$

Ideally, node n would like to set its price μ_n^k such that

$$\mu_n^k = \arg\max_{\mu_n \geq 0} R_n^k(\mu_n).$$

However, this would again require the exact knowledge of the relay traffic function $I_n(\mu_n, \mu_{-n}^{k-1})$. Alternatively, node n could try to update the price μ_n^{k-1} along a direction which tends to increase R_n^k. In the following, we show that this is indeed possible. To do this, we need to compute the left derivative, and right derivative, respectively, of the function R_n^k at μ_n^{k-1} given by

$$f_{r,n}^k(\mu_n^{k-1}) = \lim_{\delta \downarrow 0} \frac{R_n^k(\mu_n^{k-1} + \delta) - R_n^k(\mu_n^{k-1})}{\delta}$$

and

$$f_{l,n}^k(\mu_n^{k-1}) = \lim_{\delta \uparrow 0} \frac{R_n^k(\mu_n^{k-1} + \delta) - R_n^k(\mu_n^{k-1})}{\delta}.$$

Note that a positive value of the right derivative indicates the node n should increase its price in order to increase its net benefit, where as a negative value of the left derivative indicates that node n should decrease its price.

We obtain the following results.

Lemma 11: Under Assumptions 1, 3, and 4, when

$$D_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right) + i_n^{k-1} > C_n,$$

then we have that

$$f_{r,n}^k(\mu_n^{k-1}) = f_{l,n}^k(\mu_n^{k-1}) = C_n - D_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right) > 0.$$

Proof: Using the argument given in the proof of Lemma 8, one can show that when

$$D_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right) + i_n^{k-1} > C_n$$

then we have that

$$R_n^k(\mu_n^{k-1}) = U_n(x_n^k) - x_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right) + C_n \mu_n^{k-1}$$

where

$$x_n = D_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right).$$

Furthermore, using Assumptions 1 3, and 4, there exists a price $\mu'_n > \mu_n^{k-1}$ such that

$$D_n\left(\mu'_n - \hat{\kappa}_n + \lambda_n^{k-1}\right) + I_n(\mu'_n, \mu_{-n}^{k-1}) > C_n.$$

This implies that the function R_n^k is differentiable in an open interval around μ_n^{k-1} and we have

$$\begin{aligned}
& f_{r,n}^k(\mu_n^{k-1}) = f_{l,n}^k(\mu_n^{k-1}) = \\
& = U'_n(x_n) D'_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) - ... \\
& \quad - D'_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1})(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) - \\
& \quad - D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + C_n \\
& = C_n - D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}),
\end{aligned}$$

where we used the fact that

$$x_n = D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1})$$

and

$$U'_n(x_n) = \mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}.$$

The lemma then follows. ■

Lemma 12: Under Assumptions 1, 3, and 4, when $\mu_n^{k-1} > \hat{\kappa}_n \geq \beta_n(\mu_{-n}^{k-1})$ and $D_n(\lambda_n^{k-1}) + i_n^{k-1} < C_n$, then we have

$$f_{l,n}^k(\mu_n^{k-1}) < 0.$$

The above lemma can be proved using a similar argument as given in the proof of Lemma 1; we omit a detailed derivation.

Lemma 13: Under Assumptions 1, 3, and 4, when $\mu_n^{k-1} > \hat{\kappa}_n \geq \beta_n(\mu_{-n}^{k-1})$, $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} < C_n$ and $D_n(\lambda_n^{k-1}) + i_n^{k-1} \geq C_n$, then we have that

$$f_{l,n}^k(\mu_n^{k-1}) < 0.$$

Proof: Using the same line or argument as in the proof for Lemma 10, one can show that when

$$D_n\left(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}\right) + i_n^{k-1} < C_n$$

and $D_n(\lambda_n^{k-1}) + i_n^{k-1} \geq C_n$, then we have

$$R_n^k(\mu_n^{k-1}) = U_n(x_n) - x_n\lambda_n^{k-1} + i_n^{k-1}\mu_n^{k-1}$$

where $x_n = C_n - i_n^{k-1}$ and $i_n^{k-1} = I_n(\mu_n^{k-1}, \mu_n^{k-1})$. It then follows that

$$\begin{aligned}
& f_{l,n}^k(\mu_n^{k-1}) \\
& \quad = -U'_n\big(C_n - I_n(\mu_n^{k-1}, \mu_{-n}^{k-1})\big) I'_{l,n}(\mu_n^{k-1}, \mu_{-n}^{k-1}) + ... \\
& \quad\quad + (\mu_n^{k-1} + \lambda_n^{k-1}) I'_{l,n}(\mu_n^{k-1}, \mu_{-n}^{k-1}) + ... \\
& \quad\quad + I_n(\mu_n^{k-1}, \mu_{-n}^{k-1}).
\end{aligned}$$

Note that when $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} < C_n$ then we have

$$U'_n\big(C_n - I_n(\mu_n^{k-1}, \mu_{-n}^{k-1})\big) < \mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}.$$

As by assumption we have that $\mu_n^{k-1} > \hat{\kappa}_n \geq \beta_n(\mu_{-n}^{k-1})$, it follows that $\hat{\kappa}_n > \kappa_{l,n}(\mu_n^{k-1}, \mu_{-n}^{k-1})$ and

$$U'_n\big(C_n - I_n(\mu_n^{k-1}, \mu_{-n}^{k-1})\big) < \mu_n^{k-1} - \kappa_{l,n}(\mu_n^{k-1}, \mu_{-n}^{k-1}) + \lambda_n^{k-1}.$$

Bandwidth Allocation:
If $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} \geq C_n$, then set
$x_n^k = D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1})$
$y_n^k = C_n - x_n^k$;
else if $D_n(\lambda_n^{k-1}) + i_n^{k-1} < C_n$, then set
$x_n^k = D_n(\lambda_n^{k-1})$
$y_n^k = i_n^{k-1}$;
else set
$x_n^k = C_n - i_n^{k-1}$
$y_n^k = i_n^{k-1}$;
Price Update:

$$\mu_n^k = \max\Big\{\hat{\kappa}_n, \mu_n^{k-1} + \alpha_n\Big(D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + ... + i_n^{k-1} - C_n\Big)\Big\}.$$

Fig. 1. Adaptation algorithm at node n.

Using this inequality in the above expression for $f_{l,n}^k(\mu_n^{k-1})$, we obtain that $f_{l,n}^k(\mu_n^{k-1}) < 0$. ■

Note that Lemma 12 and Lemma 13 imply that node n should try to pick $\hat{\kappa}_n$ such that $\hat{\kappa}_n \geq \beta_n(\mu_{n-1}^{k-1})$.

Using the above results, we obtain the following rules for updating the price μ_n^{k-1}.

(a) Never decrease μ_n^k below $\hat{\kappa}_n$ (Lemma 12, Lemma 13).
(b) When $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} > C_n$ increase the price μ_n^{k-1} (Lemma 11).
(c) When $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} < C_n$ decrease the price μ_n^{k-1} (Lemma 12 and Lemma 13).
(d) When $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} = C_n$ leave the price μ_n^{k-1} unchanged (Proposition 1).

and we arrive at the following update rule

$$\mu_n^k = \max\left\{\hat{\kappa}_n, \mu_n^{k-1} + \alpha_n d_n^k\right\},$$

where

$$d_n^k = D_n\left(\mu_n^{k-1} + \lambda_n^{k-1} - \hat{\kappa}_n\right) + i_n^{k-1} - C_n$$

and $\alpha_n > 0$ is a small step size parameter.

C. Discussion

An important question is whether the above iterative algorithm indeed can be used to find an optimal solution to the node problem given by Eq. (1). Or more precisely, suppose we keep the prices μ_{-n} at all nodes other than node n fixed. Furthermore, suppose that node n is able to perfectly estimate the value κ_n associated with the optimal solution (x^*, y_n^*, μ_n^*) for the corresponding node problem $NODE(U_n, C_n, I_n, \mu_{-n})$ (see Proposition 1) and we have $\hat{\kappa}_n = \kappa_n$. In this case, would the above iterative algorithm converges to (x_n^*, y_n^*, μ_n^*)? The next lemma states that this is indeed the case.

Lemma 14: Suppose that we keep the the prices μ_{-n} at all nodes other than node n fixed, and suppose that $e_n^k = (x_n^k, y_n^k, \mu_n^k)$, $k \geq 0$, is the allocation sequence generated by the above iterative algorithm. Under Assumptions 1 and 2, when

$$1 - \frac{\alpha_n(L_n + H_n)}{2} > 0, \qquad n \in \mathcal{N},$$

where L_n and H_n are positive constants as given in Assumption 1, and Assumption 2, respectively, then the sequence $(e_n^k; k \geq 0)$ converges to a unique allocation $\hat{e}^* = (\hat{x}_n^*, \hat{y}_n^*, \hat{\mu}_n^*)$. Furthermore, if $\hat{\kappa}_n = \kappa_n$ where κ_n is the constant associated with the optimal solution for the node problem $NODE(U_n, C_n, I_n, \mu_{-n})$ as given by Proposition 1, then the allocation $\hat{e}^*$ is the optimal solution to the node problem $NODE(U_n, C_n, I_n, \mu_{-n})$.

Lemma 14 states that when node n is able to perfectly estimate the parameter κ_n, then the above iterative algorithm converges to an optimal solution of the node problem. In the general case, the iterative algorithm can be interpreted as a way to approximate an optimal solution to the node problem. We omit a proof for Lemma 14, but consider in the next section the more general situation where all nodes simultaneously update their allocation using the above algorithm.

V. Convergence Analysis

In this section, we consider the situation where all nodes simultaneously update their allocation using the above algorithm. We use the following notation. Let $(e^k = (e_1^k, ..., e_N^k); k \geq 0)$ be the allocation sequence generated by the adaptation algorithm where $e_n^k = (x_n^k, y_n^k, \mu_n^k)$ is the allocation decision made by node n at iteration step k. In the following, we want to analyze the properties of the limit points of the sequence $(e^k; k \geq 0)$. We start by considering the sequence of price vectors $(\mu^k; k \geq 0)$ in the next subsection.

A. Properties of the Sequence $(\mu^k; k \geq 0)$

In order to analyze the price sequence $(\mu^k; k \geq 0)$, we need an explicit expression for the demand I_n as a function of the price μ_n and μ_{-n}. Here, we make the following assumption.

Assumption 5: At iteration k, $k \geq 0$, the relay traffic at node $n \in \mathcal{N}$ is given by

$$i_n^k = I_n(\mu_n^k, \mu_{-n}^k) = \sum_{m \in H(n)} D_m(\mu_m - \hat{\kappa}_m + \lambda_m). \qquad (7)$$

Naturally one would like to keep the analysis as general as possible and avoid any simplifying assumptions whenever possible. However, as mentioned earlier, it is not clear whether the exact expression of the relay traffic $I_n(\mu_n, \mu_{-n})$, $n \in \mathcal{N}$, can be obtained as (due coupling between prices and traffic loss) this would require to obtain the closed-form solution for a fixed-point of a non-linear system of equation. Using Eq. (7), we show next that the sequence $(\mu^k; k \geq 0)$ has the property that $\lim_{k\to\infty} |\mu^k - \mu^{k+1}| = 0$. To justify the assumption given by Eq (7), we show in Proposition 5 that it is "consistent" within our model: if we have that

$$\lim_{k\to\infty} |\mu^k - \mu^{k+1}| = 0,$$

then at every limit point $\hat{e}^* = (\hat{e}_1^*, ..., \hat{e}_n^*)$ of the sequence $(e^k; k \geq 0)$ we have $\hat{y}_n^* = I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*)$ and

$$\hat{x}_n^* = D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*), \qquad n \in \mathcal{N},$$

where $\hat{\lambda}_n^* = \sum_{m \in r_n} \hat{\mu}_m^*$.

Let the Lyapunov function $\Phi : \Re_+^N \to \Re_+$ be given by

$$\Phi(\mu) = \sum_{n \in \mathcal{N}} \left[\int_{\zeta=0}^{\mu_n + \lambda_n} D_n(\zeta - \hat{\kappa}_n) d\zeta - \mu_n C_n \right],$$

where $\lambda_n = \sum_{m \in r_n} \mu_m$. We have the following result.

Lemma 15: Under Assumption 2, there exists a constant $L > 0$ such that

$$||\nabla\Phi(\mu) - \nabla\Phi(\eta)|| \leq L||\mu - \eta||, \quad \mu, \eta \in \Re_+^N.$$

The above lemma follows from Assumption 2 which establishes that the demand function D_n at every node $n \in \mathcal{N}$ satisfies a Lipschitz condition. We omit here a detailed derivation.

Proposition 4: Under Assumptions 1, 2, and 5, when

$$1 - \frac{\alpha_n L}{2} > 0, \qquad n \in \mathcal{N},$$

where L is the positive constant of Lemma 15, then we have that $\lim_{k\to\infty} |\mu^k - \mu^{k+1}| = 0$.

The above proposition establishes that the change in the price vector μ^k approaches zero given that all nodes use a small enough step size parameter. Note however that the proposition does not imply that the sequence $(\mu^k, k \geq 1)$ converges. Using the Lyapunov function $\Phi(\mu)$, Proposition 4 can be proved using the same line of argument as given by Kelly *et al.* in [7]. We omit here a detailed derivation.

B. Properties of Limits Points

Let $E = \{e^k, k \geq 0\}$ be the set of allocation vectors generated by the iterative algorithm. Set E has the following properties.

Lemma 16: Under Assumptions 1 and 2, there exists a constant $B > 0$ such that $||e^k|| < B, \quad k \geq 1$.

The above lemma states that the set E is bounded. This result follows immediately from Assumption 1 and the definition of the iterative algorithm given in Fig. 1. We omit a detailed derivation.

Lemma 17: Every infinite subset of E has a limit point.

The above lemma follows immediately from Lemma 16 (see for example [13]).

Let E^* be the set of all limit points of E. Using the result of Proposition 4 that $\lim_{k\to\infty} |\mu_n^k - \mu_n^{k+1}| = 0, n \in \mathcal{N}$, we obtain the following result.

Proposition 5: Under Assumptions 1 and 2, if

$$\lim_{k\to\infty} |\mu_n^k - \mu_n^{k+1}| = 0$$

then the sequence $(e^k, k \geq 1)$ converges to the set E^*. Furthermore, for every $\hat{e}^* \in E^*$ we have that $\hat{\mu}_n^* \geq \hat{\kappa}_n$ and

$$\begin{aligned} \hat{x}_n^* &= D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*) \\ \hat{y}_n^* &= I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*) \\ \hat{\mu}_n^* &= \hat{\kappa}_n, \qquad \text{if } \hat{x}_n^* + \hat{y}_n^* < C_n, \end{aligned}$$

where $\hat{\lambda}_n^* = \sum_{m \in r_n} \hat{\mu}_m^*$.

We provide a proof of Proposition 5 in the appendix. In order to prove Proposition 5, we use from Subsection V-A only the result of Proposition 4 that $\lim_{k\to\infty}|\mu_n^k - \mu_n^{k+1}| = 0, n \in \mathcal{N}$, and we do not explicitly use the assumption that the relay traffic is given by Eq.(7).

Note that the limit points of E have very similar to the properties as a Nash equilibrium of the network market (see Proposition 2). Combining this observation with Lemma 3, we then obtain the following convergence results.

Corollary 1: Under Assumptions 1 and 2, if

$$\lim_{k\to\infty}|\mu_n^k - \mu_n^{k+1}| = 0$$

then the sequence $(x^k, k \geq 1)$ converges to the unique solution of the maximization problem

$$\max_{x\in\Re_+^N} \sum_{n\in\mathcal{N}} \left(U_n(x_n) - x_n \sum_{m\in r_n} \hat{\kappa}_m \right)$$
$$\text{subject to} \qquad Ax \leq C.$$

VI. Estimating $\hat{\kappa}_n$

In this section, we address the question how node n should choose its estimate $\hat{\kappa}_n$ in order to maximize its net benefit. We use the following definition.

Definition 2: Let the estimates $\hat{\kappa} = (\hat{\kappa}_1, ..., \hat{\kappa}_N)$ be given and consider the sequence $(e^k, k \geq 1)$ generated using $\hat{\kappa}$. We call $\hat{\kappa}$ an optimal estimate vector if for every limit point $\hat{e}^*$ of the sequence $(e^k; k \geq 0)$ we have that for every node n the allocation $(\hat{x}_n^*, \hat{y}_n^*, \hat{\mu}_n^*)$ is an optimal solution to the node problem $NODE(U_n, C_n, I_n, \hat{\mu}_{-n}^*)$.

The above definition implies that under an optimal estimate vector $\hat{\kappa}$, the above iterative algorithm result in an allocation that maximizes the net benefit of each node, *i.e.* solves the node problem of Section II for each node n.

We have the following result.

Corollary 2: Under Assumptions 1 and 2, if

$$\lim_{k\to\infty}|\mu_n^k - \mu_n^{k+1}| = 0$$

then a vector $\hat{\kappa} = (\hat{\kappa}_1, ..., \hat{\kappa}_N)$ is an optimal estimate vector if and only if there exists a market equilibrium e^* such that for every node n we have $\kappa_{r,n}(\mu_n^*, \mu_{-n}^*) \leq \hat{\kappa}_n \leq \kappa_{l,n}(\mu_n^*, \mu_{-n}^*)$, $\mu_n^* \geq \hat{\kappa}_n$, and

$$\begin{aligned} x_n^* &= D_n\Big(\mu_n^* - \hat{\kappa}_n + \lambda_n^*\Big) \\ y_n^* &= I_n(\mu_n^*, \mu_{-n}^*) \\ \mu_n^* &= \hat{\kappa}_n, \qquad \text{if } x_n^* + y_n^* < C_n. \end{aligned}$$

The above result follows immediately from Proposition 2 and 5.

Corollary 2 states that if nodes have perfect information about the function I_n and therefore can perfectly estimate the vector $\kappa = (\kappa_1, ..., \kappa_N)$ of a market equilibrium as given in Section III, then the adaptation mechanism of Section IV indeed converges to a market equilibrium. When nodes do not have perfect information (as it will be the case in practice), we can interpret the iterative algorithm of Section IV will approximate a market equilibrium as defined in Section III.

VII. Numerical Results

We illustrate the iterative algorithm given in Fig. 1 of Section IV using a numerical case study. In particular, we wish to verify whether the price sequence $(\mu^k; k \geq 0)$ indeed has the property that $\lim_{k\to\infty}|\mu_n^k - \mu_n^{k+1}| = 0$ as predicted by Proposition 4.

We consider a network consisting of 25 nodes and fixed routes as given in Fig. 2. The capacity at every node is set equal to 1 (unit traffic/second) and the demand function at each node $n \in \mathcal{N}$ is given by

$$D_n(\lambda_n) = e^{-\lambda_n}, \qquad \lambda_n \geq 0.$$

In addition, we assume that each node sets its estimate $\hat{\kappa}_n$ equal to 0.1. Setting $\alpha_n = 0.1$, $n \in \mathcal{N}$, we simulated the system for 200 iteration steps.

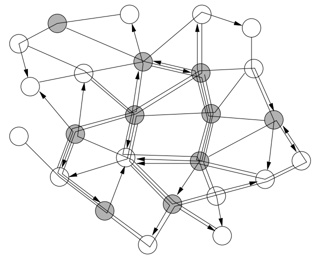

Fig. 2. Network Traffic Flows

Figure 3, and Figure 4, indicate the trajectories of the transmission rates x_n, and the price μ_n, respectively, at individual node $n \in \mathcal{N}$. We note that if node n has spare capacity in equilibrium then the price converges to 0.1 (the value of $\hat{\kappa}_n$). In Fig. 2, we highlighted the nodes for which converge to a price higher than 0.1.

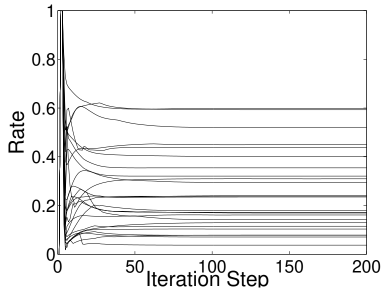

Fig. 3. Trajectories of the transmission rates x_n.

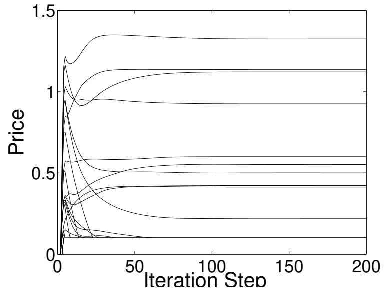

Fig. 4. Trajectories of the price μ_n.

We note that in the above simulation we have that the price changes converge to 0 as predicted by Proposition 4. We carried out additional simulations with different topologies and demand functions, and for all experiments the price changes converged to 0 (given that the step-size parameter is chosen small enough). This indicates that the analysis of Section V-A provides the correct insight about the convergence behavior of the algorithm, even though it uses the simplifying assumption on the relay traffic at each node.

VIII. Battery Cost

In our discussion so far, we ignored the fact the nodes in a wireless ad hoc network may have limited battery resources. In this section, we outline how this can be incorporated into our model.

A node with a low battery will be reluctant to relay packets for other nodes as this might impact the node's ability to send its own traffic. Moreover, node n will also carefully manage how it uses the remaining battery power to send its own traffic. In particular, it will only send traffic that has a high value for node n. We can model this situation through an opportunity cost p_n which captures the trade-off between using battery to obtain an instantaneous utility/reward and saving battery to obtain a potentially higher utility/reward in the future. The cost p_n will be higher at nodes which are currently low on battery. We assume that the cost p_n changes slowly compared with the time-scale at which nodes update their bandwidth allocation (x_n, y_n) and the price μ_n.

Let p_n be the battery cost of node n. As node n has to use power for receiving and sending relay traffic, we associate the following maximization problem with node n,

$$\max_{x_n, y_n, \mu_n} \Big\{ U_n(x_n) - x_n\lambda_n + y_n\mu_n - p_n(x_n + 2y_n) \Big\}.$$

$$\begin{aligned} \text{subject to} \qquad & x_n + y_n \leq C_n \\ & y_n \leq I_n(\mu_n, \mu_{-n}) \\ & x_n, y_n, \mu_n \geq 0. \end{aligned}$$

Note that this problem has the same structure as the maximization problem $NODE(U_n, C_n, I_n, \mu_{-n})$, except that it also accounts for opportunity cost of depleting the battery at rate $x_n + 2y_n$. The analysis presented in the previous sections can then easily be extended to this situation, requiring only notational changes. The modified iterative algorithm is given in Fig. 5 and it can be shown that algorithm converges to a bandwidth allocation which maximizes the following objective function

$$\max_{x \in \Re_+^N} \sum_{n \in \mathcal{N}} \left(U_r(x_n) - x_n \left(p_n + \sum_{m \in r_n} (\hat{\kappa}_m + 2p_m) \right) \right)$$

$$\text{subject to} \qquad Ax \leq C.$$

Compared with the bandwidth allocation obtained in Corollary 1, node n will reduce its transmission rate if (a) its own battery is low (and p_n is large) or (b) its route passes through another node with a low battery (and $\sum_{m \in r_n} 2p_m$ is large).

Bandwidth Allocation:
If $D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1} + p_n) + i_n^{k-1} \geq C_n$, then set
$x_n^k = D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1} + p_n)$
$y_n^k = C_n - x_n^k$;
else if $D_n(\lambda_n^{k-1} + p_n) + i_n^{k-1} < C_n$, then set
$x_n^k = D_n(\lambda_n^{k-1} + p_n)$
$y_n^k = i_n^{k-1}$;
else set
$x_n^k = C_n - i_n^{k-1}$
$y_n^k = i_n^{k-1}$;
Price Update:

$$\mu_n^k = 2p_n + \alpha_n \Big[\mu_n^{k-1} + \ldots + \Big(D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1} + p_n) + i_n^{k-1} - C_n \Big) \Big]^+.$$

Fig. 5. Iterative algorithm with battery cost p_n.

IX. Conclusions

We analyzed a market-based mechanism for stimulating cooperation in wireless ad -hoc networks, where each node can freely decide on the amount of traffic it relays and how much it charges other nodes for relaying their traffic. We also discussed how battery constrains can be modeled. Our analysis adds to earlier work in [6], [12] by formally deriving how a selfish node would adapt its bandwidth allocation and price.

To formulate the network model, we assumed that lower layers determine the routes along which traffic is sent, as well as the total transmission rate that is allocated to each node. For future work, it would be desirable to revisit this assumption and investigate how the proposed price-based scheme interacts with routing and bandwidth allocation. For example, nodes might want to send their traffic along the cheapest route, leading to a coupling between routing and pricing. Similarly, there is a coupling between the price-based mechanism and the bandwidth allocation decision made by lower layers protocols. For example, when a node does not fully use its allocated bandwidth then lower layer protocols may want to reallocated the spare capacity to other nodes in order to increase the overall efficiency of the system. Modeling and analyzing these cross-layer interaction seems possible, but poses additional challenges that have to be carefully studied.

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Appendix

A. *Proof of Proposition 1*

To prove Proposition 1, we first establish few preliminary lemmas.

Lemma 18: If $D_n(\lambda_n) + I_n(\beta_n(\mu_{-n}), \mu_{-n}) < C_n$, then the unique optimal solution to the node problem is given $\mu_n^* = \beta_n(\mu_{-n}^*)$, $x_n^* = D_n(\lambda_n)$ and $y_n^* = I_n(\mu_n^*, \mu_{-n})$.
The proof of the above lemma is similar to the proof of Lemma 1 and we omit here a detailed derivation.

Lemma 19: Consider an allocation (x_n, y_n, μ_n) such that $x_n + y_n = C_n$. If we have that

$$D_n(\mu_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \lambda_n) + I_n(\mu_n, \mu_{-n}) < C_n,$$

then (x_n, y_n, μ_n) is not an optimal allocation to the node problem given by Eq. (1).

Proof: Assume the contrary and suppose that the allocation (x_n, y_n, μ_n) with $x_n + y_n = C_n$ and

$$D_n(\mu_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \lambda_n) + I_n(\mu_n, \mu_{-n}) < C_n,$$

is an optimal allocation. If this is the case, then by Lemma 2 we have that $\mu_n \geq \beta_n(\mu_{-n})$ and $y_n = I_n(\mu_n, \mu_{-n})$. Consider now the function

$$R(p) = U_n(x_n) - \lambda_n x_n + p I_n(p, \mu_{-n}),$$

where $x_n = C_n - I_n(p, \mu_{-n})$. Note that $R(\mu_n)$ is the net benefit at node n for the above allocation (x_n, y_n, μ_n). We then have that

$$\lim_{\delta \uparrow 0} \frac{R(\mu_n + \delta) - R(\mu_n)}{\delta} = I'_{l,n}(\mu_n, \mu_{-n})\Big[- U'_n(x_n) + \lambda_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \mu_n \Big].$$

As we have $D_n(\mu_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \lambda_n) + I_n(\mu_n, \mu_{-n}) < C_n$ and $x_n = C_n - I_n(\mu_n, \mu_{-n}$, it follows that

$$x_n > D_n(\mu_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \lambda_n)$$

and

$$U'(x_n) < \mu_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \lambda_n.$$

Combining the above results, it follows that

$$\lim_{\delta \uparrow 0} \frac{R(\mu_n + \delta) - R(\mu_n)}{\delta} < 0,$$

which implies that an allocation (x_n, y_n, μ_n) is not an optimal allocation. ■

Lemma 20: Consider an allocation (x_n, y_n, μ_n) such that $x_n + y_n = C_n$, $\mu_n \geq \beta_n(\mu_{-n})$ and $y_n = I_n(\mu_n, \mu_{-n})$. If we have that

$$D_n(\mu_n - \kappa_{r,n}(\mu_n, \mu_{-n}) + \lambda_n) + I_n(\mu_n, \mu_{-n}) > C_n,$$

then (x_n, y_n, μ_n) is not an optimal allocation to the node problem given by Eq. (1).
The above lemma can be proved using a similar argument as given for Lemma 19, we omit here a detailed derivation.

Lemma 21: If $D_n(\lambda_n) + I_n(\beta_n(\mu_{-n}), \mu_{-n}) \geq C_n$, then the unique optimal solution to the node problem is given $y_n^* = I_n(\mu_n^*, \mu_{-n})$ and $x_n^* = C_n - y_n^*$, where μ_u^* is the unique price such that

$$D_n(\mu_n^* - \kappa_{l,n}(\mu_n^*, \mu_{-n}) + \lambda_n) + I_n(\mu_n^*, \mu_{-n}) \geq C_n$$

and

$$D_n(\mu_n^* - \kappa_{r,n}(\mu_n^*, \mu_{-n}) + \lambda_n) + I_n(\mu_n^*, \mu_{-n}) \leq C_n.$$

Proof: Using Assumption 1, Assumption 5, Assumption 4, and Lemma 1, it follows that there exists a unique price μ_n^* such that

$$D_n(\mu_n - \kappa_{l,n}(\mu_n, \mu_{-n}) + \lambda_n) + I_n(\mu_n, \mu_{-n}) \geq C_n$$

and

$$D_n(\mu_n - \kappa_{r,n}(\mu_n, \mu_{-n}) + \lambda_n) + I_n(\mu_n, \mu_{-n}) \leq C_n;$$

furthermore we have that $\mu_n^* \geq \beta_n(\mu_{-n})$.

Next, we show that for any allocation $(\bar{x}_n, \bar{y}_n, \bar{\mu}_n)$, $(\bar{x}_n, \bar{y}_n, \bar{\mu}_n) \neq (x_n^*, y_n^*, \mu_n^*)$, with $\bar{x}_n + \bar{y}_n \leq C_n$, we have that

$$U_n(\bar{x}_n) - \bar{x}_n\lambda_n + \bar{\mu}_n \min\{\bar{y}_n, I_n(\bar{\mu}_n, \mu_{-n})\} < U_n(x_n^*) - x_n^*\lambda_n + \mu_n^* y_n^*.$$

Let us first consider such allocations $(\bar{x}_n, \bar{y}_n, \bar{\mu}_n)$ such that $\bar{x}_n + \bar{y}_n = C_n$. Using Lemma 2, it suffices to consider allocation such that $y_n = I_n(\bar{\mu}_n, \mu_{-n})$. In this case, we have to show that

$$U_n(\bar{x}_n) - \bar{x}_n\lambda_n + \bar{\mu}_n I_n(\bar{\mu}_n, \mu_{-n}) < U_n(x_n^*) - x_n^*\lambda_n + \mu_n^* y_n^*,$$

where $\bar{x}_n = C_n - I_n(\bar{\mu}_n, \mu_{-n})$. It can be shown that this is indeed true using a similar argument as given in the proof of Lemma 19.

Next, we consider allocations $(\bar{x}_n, \bar{y}_n, \bar{\mu}_n)$ such that $\bar{x}_n + \bar{y}_n < C_n$. In this case, let the price $\bar{\mu}'_n$ be such that

$$\bar{\mu}'_n = \min\{\mu_n \geq \beta_n(\mu_{-n}) | I_n(\mu_n, \mu_{-n}) + \bar{x}_n \leq C_n\}.$$

If $I_n(\bar{\mu}'_n, \mu_{-n}) + \bar{x}_n \leq C_n$, then by Lemma 1 we have for $\bar{y}'_n = I_n(\bar{\mu}'_n, \mu_{-n})$ and $\bar{x}'_n = \bar{x}_n = C_n - \bar{y}'_n$ that

$$U_n(\bar{x}_n) - \bar{x}_n\lambda_n + \bar{y}_n < U_n(\bar{x}'_n) - \bar{x}'_n\lambda_n + \mu_n\bar{y}'_n.$$

Similarly, if $I_n(\bar{\mu}'_n, \mu_{-n}) + \bar{x}_n \leq C_n$, then we have for $\bar{\mu}'_n = \beta_n(\mu_{-n})$, $\bar{y}'_n = I_n(\bar{\mu}'_n, \mu_{-n})$, and $\bar{x}'_n = C_n - \bar{y}'_n$ that

$$U_n(\bar{x}_n) - \bar{x}_n\lambda_n + \bar{y}_n < U_n(\bar{x}'_n) - \bar{x}'_n\lambda_n + \mu_n\bar{y}'_n.$$

Therefore, if $(\bar{x}_n, \bar{y}_n, \bar{\mu}_n)$ such that $\bar{x}_n + \bar{y}_n < C_n$ there always exists an allocation $(\bar{x}'_n, \bar{y}'_n, \bar{\mu}'_n)$ with $\bar{x}'_n + \bar{y}'_n = C_n$ such that

$$U_n(\bar{x}_n) - \bar{x}_n\lambda_n + \bar{y}_n < U_n(\bar{x}'_n) - \bar{x}'_n\lambda_n + \mu_n\bar{y}'_n.$$

Hence, using the discussion above, we have that

$$U_n(\bar{x}_n) - \bar{x}_n\lambda_n + \bar{y}_n < U_n(x^*_n) - x^*_n\lambda_n + \mu^*_n y^*_n.$$

and the lemma follows. ■

Proposition 1 then follows immediately from Lemma 18 and Lemma 21.

B. Proof of Proposition 3

We first prove Proposition 3 assuming that for every node $n \in \mathcal{N}$ the relay function $I_n(\mu_n, \mu_{-n})$ is continuously differentiable.

Note that in order to establish that there exists a Nash equilibrium in this case, it suffices to consider the maximization problem

$$\begin{array}{ll} \max\limits_{x \in \Re^N_+} \sum\limits_{n \in \mathcal{N}} \left(U_n(x_n) - x_n \sum\limits_{m \in r_n} \kappa_m(\mu^*_n, \mu^*_{-n}) \right) \\ \text{subject to} \qquad Ax \leq C \end{array}$$

and verify that there exists a price vector $(\mu^*_1, ..., \mu^*_n) \in \Re^N_+$ such that for the Lagrange multipliers $(\theta^*_1, ..., \theta^*_n)$ in the proof of Lemma 3, we have that $\mu^*_n = \theta^*_n + \kappa_n(\mu^*_n, \mu^*_{-n})$. Let the price λ_{max} be such that

$$0 < \sum_{m \in H(n)} D_m(\lambda_{max}) \leq C_n, \qquad n \in \mathcal{N},$$

and let κ_{max} be such that for all nodes $n \in \mathcal{N}$ we have that

$$\kappa_{max} \geq \kappa_n(\mu_n, \mu_{-n}), \qquad (\mu_n, \mu_{-n}) \in \Re^N_+.$$

Note that by Assumption 1, 3, and 4, we have that λ_{max} and $\kappa_{\max}$ are finite. Furthermore, the Lagrange multipliers $(\theta^*_1, ..., \theta^*_n)$ in the proof of Lemma 3 are such that

$$\sum_{m \in H(n)} D_m \left(\theta^*_m + \sum_{m' \in r_m} \left(\theta^*_{m'} + \kappa_{m'}(\mu^*_{m'}, \mu^*_{-m'}) \right) \right) \leq C_n,$$

and θ^*_n is equal to 0 if

$$\sum_{m \in H(n)} D_m \left(\theta^*_m + \sum_{m' \in r_m} \left(\theta^*_{m'} + \kappa_{m'}(\mu^*_{m'}, \mu^*_{-m'}) \right) \right) < C_n.$$

Furthermore, recall that for the optimal bandwidth allocation $x^* = (x^*_1, ..., x^*_N)$ to the maximization problem given in Eq. (3) we have

$$x^*_n = D_n \left(\theta^*_n + \sum_{m \in r_n} \left(\theta^*_m + \kappa_m(\mu^*_m, \mu^*_{-m}) \right) \right), \quad n \in \mathcal{N}.$$

It then follows that $\theta^*_n \leq \lambda_{max}$, $n \in \mathcal{N}$. Let the set $\mathcal{B}$ be given by $\mathcal{B} = [0, \lambda_{max} + \kappa_{max}]^N$ and consider the mapping $f : \mathcal{B} \mapsto \mathcal{B}$ for which the nth component is given by

$$f_n(\mu) = \theta^*_n(\mu) + \kappa_n(\mu_n, \mu_{-n}), \qquad n = 1, ..., N,$$

where $\theta^*_n(\mu)$ is an optimal Lagrange multiplier for the capacity constraint associated with node n in the optimization problem

$$\begin{array}{ll} \max\limits_{x \in \Re^+_n} \sum\limits_{n \in \mathcal{N}} \left(U_n(x_n) - x_n \sum\limits_{m \in r_n} \kappa_m(\mu_m, \mu_{-m}) \right) \\ \text{subject to} \qquad Ax \leq C. \end{array}$$

Note that by Assumption 4, we have that $\kappa_n(\mu_n, \mu_{-n})$ is continuous in $\mu \in \mathcal{B}$. Furthermore, using sensitivity analysis for Lagrange multipliers (see for example [1]), the function $\theta^*(\mu) = (\theta^*_1(\mu), ..., \theta^*_N(\mu))$ can be chosen to be continuous on $\mathcal{B}$. As a result, we have that the function $f(\mu)$ can be chosen to be continuous mapping of $\mathcal{B}$ on itself. As the set $\mathcal{B}$ is convex and compact, by Brouwer's fixed point theorem there exists a vector $\mu^* \in \mathcal{B}$ such that

$$\mu^*_n = f_n(\mu^*) = \theta^*_n(\mu^*) - \kappa_n(\mu^*_n, \mu^*_{-n}), \qquad n \in \mathcal{N}.$$

The proposition then follows.

If the functions characterizing the relay traffic at individual nodes is not everywhere continuously differentiable, but has the properties as given in Assumption 5, then the function $f(u)$ that we defined above is not necessarily continuous. However, under Assumption 5 it can be shown that $f(u)$ can be chosen to be a upper semi-continuous correspondence and the existence of a Nash equilibrium follows from Kakutani's fixed-point theorem [2]. We omit here a detailed derivation.

C. Proof of Proposition 5

To study the properties of a limit point $\hat{e}^*$ of E, we proceed as follows. Given a node $n \in \mathcal{N}$ and an allocation vector e, we consider the function

$$\begin{aligned} f_n(e) \quad = \quad & \Big(x_n - D_n(\mu_n - \hat{\kappa}_n + \lambda_n)\Big)^2 + ... \\ & + \Big(y_n - I_n(\mu_n, \mu_{-n})\Big)^2 + ... \\ & + (\mu_n - \hat{\kappa}_n)(C_n - x_n - y_n). \end{aligned}$$

For a given limit point $\hat{e}^*$ of E, let $(e^{k_i}; i \geq 0)$ be the subsequence that converges to $\hat{e}^*_n$. As the function f_n is continuous, it then follows that for every node n we have we have

$$\lim_{i \to \infty} f_n(e^{k_i}) = f_n(\hat{e}^*).$$

Using this result, we show in the following that for every limit point $\hat{e}^*$ and every node $n \in \mathcal{N}$, we have that

$$f_n(\hat{e}^*) = 0. \tag{8}$$

The above results then implies Proposition 5.

Proposition 6: For every limit point $\hat{e}^*$ of E we have

$$f_n(\hat{e}^*) = 0, \qquad n \in \mathcal{N}.$$

Proof: We use the following notation. Let d_n^k be the price update direction at step k, *i.e.* we have

$$d_n^k = D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) - i_n^{k-1} - C_n,$$

and let $\hat{d}_n^k$ be the projection of d_n^k to ensure that the price μ_n^k is non-negative given by

$$\hat{d}_n^k = \begin{cases} d_n^k & \text{if } \mu_n^{k-1} + \alpha_n d_n^k \geq \hat{\kappa}_n, \\ \frac{-\mu_n^{k-1} + \hat{\kappa}_n}{\alpha_n} & \text{if } \mu_n^{k-1} + \alpha_n d_n^k < \hat{\kappa}_n. \end{cases}$$

By assumption we have that $\lim_{k \to \infty} \hat{d}_n^k = 0$.

Let J be the set of non-negative integers and consider the following two subsets of J,

$$J_1 = \left\{ k \geq 0 \mid \hat{d}_n^k = d_n^k \right\}$$

and

$$J_2 = \left\{ k \geq 0 \mid \hat{d}_n^k = \frac{-\mu_n^{k-1} + \kappa_n}{\alpha_n} \right\}.$$

Note that $J = J_1 \cup J_2$. As the set J is infinite, we have at least one of the two subsets J_1 and J_2 is infinite.

Suppose that the set J_1 is infinite. Combining the definition of the set J_1 with Proposition 4, we have for every limit point $\hat{e}^*$ of the sequence $(e^k; k \in J)$ that

$$D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*) - I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*) - C_n = 0, \qquad n \in \mathcal{N}.$$

Consider now the following three subsets of J_1,

$$\begin{aligned} J_1^1 &= \{k \in J_1 \mid D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} \geq C_n\}, \\ J_1^2 &= \{k \in J_1 \mid D_n(\lambda_n^{k-1}) + i_n^{k-1} < C_n\}, \end{aligned}$$

and

$$\begin{aligned} J_1^3 = \{k \in J_1 \quad | \quad & D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + i_n^{k-1} \geq C_n \\ & \text{and } D_n(\lambda_n^{k-1}) + i_n^{k-1} \geq C_n\}. \end{aligned}$$

Note that $J_1 = J_1^1 \cup J_1^2 \cup J_1^3$. It follows that when the set J_1 is infinite, then at least one of these three sets of J_1 is infinite. Suppose that the set J_1^1 is infinite. Combining the definition of the set J_1^1 with the update rules given in Fig. 1, for every $k \in J_1^1$ we have $x_n^k = D_n(\mu_n^k - \hat{\kappa}_n + \lambda_n^k)$ and $y_n^k = C_n - x_n^k$. Combining the above results, it follows that for every limit $\hat{e}^*$ point of the sequence $(e^k; k \in J_1^1)$ we have that

$$D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*) - I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*) - C_n = 0$$

and

$$\hat{y}_n^* = I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*).$$

Hence, we obtain that at each limit point $\hat{e}^*$ of of the sequence $(e^k; k \in J_1^1)$ we have that $f_n(\hat{e}^*) = 0$.

Similarly, when the set J_1^2 is infinite, then we have for every limit $\hat{e}^*$ point of the sequence $(e^k; k \in J_1^2)$ that $\hat{x}_n^* = D_n(\hat{\lambda}_n^*)$ and $\hat{y}_n^* = I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*)$. Note that for $k \in J_1^2$ we have that

$$\begin{aligned} & D_n(\mu_n^{k-1} - \hat{\kappa}_n + \lambda_n^{k-1}) + I_n(\mu_n^{k-1}, \mu_{-n}^{k-1}) - C_n \\ & \leq D_n(\lambda_n^{k-1}) + I_n(\mu_n^{k-1}, \mu_{-n}^{k-1}) - C_n \\ & < 0. \end{aligned}$$

As we have that

$$\lim_{k \to \infty : k \in J_1^2} D_n(\mu_n^{k-1} - \hat{\kappa}_n - \lambda_n^{k-1}) + I_n(\mu_n^{k-1}, \mu_{-n}^{k-1}) - C_n = 0,$$

it follows that at every limit point $\hat{e}_n^*$ of the sequence $(e^k; k \in J_1^2)$ we have

$$D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*) = D_n(\hat{\lambda}_n^*) = C_n - I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*).$$

Combining this result with the fact that $\hat{x}_n^* = D_n(\hat{\lambda}_n^*)$ and $\hat{y}_n^* = I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*)$, it follows that at every limit point $\hat{e}^*$ of the sequence $(e^k; k \in J_1^2)$ we have that $f_n(\hat{e}^*) = 0$.

Finally, suppose that the set J_1^3 is infinite. For every limit point $\hat{e}^*$ of $(e^k; k \in J_1^3)$, we have that $\hat{y}_n^* = I_n(\hat{\mu}_n^*, \hat{\mu}_{-n}^*)$ and $\hat{x}_n^* = C_n - \hat{y}_n^*$, and that $D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*) + I_n(\hat{\mu}_n^*, \mu_{-n}^* - C_n = 0$. Combining these results, we obtain that $\hat{x}_n^* = D_n(\hat{\mu}_n^* - \hat{\kappa}_n + \hat{\lambda}_n^*)$ and $f_n(\hat{e}^*) = 0$.

The above analysis shows that for every limit point $\hat{e}^*$ of $(e^k; k \in J_1)$ we have that $f_n(\hat{e}^*) = 0$.

Using a similar analysis as given above for the set J_1, one can show that for every limit point $\hat{e}^*$ of $(e^k; k \in J_2)$ we have that $f_n(\hat{e}^*) = 0$. It follows that for every limit point $\hat{e}^*$ of $(e^k; k \geq 0)$ we have that $f_n(\hat{e}^*) = 0$.

The result that the sequence $\{e^k\}_{k \geq 0}$ converges to the set E^* follows from the fact that

$$\lim_{k \to \infty} |\mu_n^k - \mu_n^{k+1}| = 0$$

and the assumption that the functions D_n and I_n are continuous. ■

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