Analysis of a Static Pricing Scheme for Priority Services

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Abstract— **We analyze a static pricing scheme for priority services. Users are free to choose the priority of their traffic but are charged accordingly. Using a game theoretic framework we study the case where users choose priorities to maximize their net benefit. For the single link case, we show that there always exists an equilibrium for the corresponding game; however, the equilibrium is not necessarily unique. Furthermore, we show that packet loss in equilibrium can be expressed as a function of the prices associated with the different priority classes. We provide a numerical case study to illustrate our results.**

Keywords— **Priority service, pricing, congestion control, differentiated services.**

I. Introduction

Today's "best effort" Internet is enough to support services such as web browsing and e-mail, but it is not sufficient for new real-time services such as voice and video which require a more predictable Quality of Service (QoS) model. One proposed means to address this issue is to classify Internet traffic into priority classes (see Diffserv proposal from the IETF [9]). Clearly, networks offering this service model need a mechanism to allocate traffic to priority classes; otherwise all users declare their traffic as high priority and the above priority model degenerates to a best-effort service. Here, we consider a price-based allocation scheme. We make the following assumptions regarding priority assignment, pricing, and charging.

We assume that users assign packets to priority classes and that the priority of a packet does not change as it travels through the network.

We consider a static pricing scheme where prices associated with the different priority classes are fixed and do not change as the traffic load in the network changes. Static pricing schemes are simpler to implement than dynamic schemes which adapt prices as the traffic load changes. However, as prices are chosen a priori without any detailed knowledge of the actual traffic demand, it is not clear how static pricing can be used to achieve a certain performance objective, or maximize network revenue. One goal of this paper is to obtain a better understanding of this issue.

Finally, we assume that users are charged for packets submitted to the network rather than for packets actually delivered. This simplifies the implementation as there is no need to track dropped packets for accounting and billing purposes. Furthermore, our analysis reveals that charging for submitted traffic encourages users to reduce their transmission rate when network congestion occurs. We will discuss this property in Section IV and show how it can be used as a congestion control mechanism.

We study the following questions for the above pricing scheme. *Stability:* Does this scheme lead to a stable operation of the network? *Bandwidth Allocation:* How is the network bandwidth shared among users? *Revenue:* How should prices be chosen to maximize revenue? *QoS:* How do pricing decision affect the QoS provided by the network?

For our analysis, we focus on the case where users measure the network performance (QoS) as a function of their average throughput and characterize users' bandwidth requirements by an utility function. Roughly, submitting more high priority traffic will increase the utility of an individual user. However, doing so will also increase the cost that users have to pay to the network. We model this trade-off as an optimization problem where users choose priorities to maximize their net benefit. We then consider the non-cooperative game [5] where all users simultaneously try to maximize their net benefit. To formulate an equilibrium for this game, we assume that changes in the traffic allocation of an individual user has a negligible effect on the overall network. We note that a similar assumption is used in economic theory to define the Walras equilibrium for large economies [28] and in transportation networks to define a Wardrop equilibrium ([8], [30]). We will comment in some more details on this assumption and its relaxation in Section II.

For the single link case, we obtain the following answers for the above questions.

Stability: We show that there always exists a (Wardrop) equilibrium allocation; however, the equilibrium is not necessarily unique.

Bandwidth Allocation: A somewhat surprising result of our analysis shows that the equilibrium bandwidth allocation does not depend on the prices of the different priority classes. For the case of a homogeneous user population (*i.e.* all users have the same utility function), bandwidth is equally shared among users. For the general case of a heterogeneous user population, there exists a unique clearance price which only depends on the total traffic demand of all users (and is independent of price choices of the different priority classes) and the bandwidth allocated to a user is equal to its demand under this clearance price (see Section IV-B).

Revenue: A consequence of the above result is that price decisions do not affect network revenue (see Section IV-C).

Quality of Service: However, prices do affect the QoS in the different priority classes. In particular, we show that there exists a simple relation between prices and packet

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loss. This result suggest that static pricing can be used as a tool for congestion control (see Section IV-D).

The main contributions of our paper are as follows. We extend earlier work that is restricted to two priority classes [15], or to an idealized priority service which support a continuum of priorities [19]. Our results clarify the role of price decisions. We show that prices do not influence the bandwidth allocation or network revenue; however pricing can be used to guarantee a minimal QoS in terms of packets loss.

Below, we provide an overview of price-based mechanism for multi-class QoS provision and their game-theoretic representation ([1], [6], [7], [10], [13], [14], [15], [20], [23], [25], [26], [27], [29]). These models differ along several dimensions, some of the important ones being (a) packet scheduling routines, (b) shape of the users' utility function, (c) pricing mechanisms, and (d) whether users are allowed to split their traffic among several service classes. We use these four criteria to compare the above approaches to multi-class QoS provision.

Packet Scheduling: Multi-class QoS provisioning relies on packet scheduling mechanisms which implement a class-dependent packet-forwarding rule. Examples of such schemes are strict priority scheduling (SP) and generalized processor sharing (GPS), also known as weighted fair queueing (WFQ) ([3], [24]). SP inherently leads to an ordering of the QoS in the different classes; however, it is well-known that SP can lead to a two-level service which does not allow graded multi-class QoS provisioning (we comment on that in more detail in Section IV-E). On the other hand, GPS/WFQ allows multi-class QoS provisioning with graded performance across all services classes; however, GPS/WFQ with fixed weights has no inherent service class ordering which complicates both analysis and service provisioning. In [26], Ren and Park propose and analyze a general class of schedulers for multi-class QoS provisioning which contains both GPS/WFQ and SP. The optimal scheduler derived by Ren and Park can be represented by a GPS/WFQ scheduler with variable weights. It is shown that this scheduler combines the advantages of GPS/WFQ and SP scheduling and provides both graded multi-class QoS and service ordering. In this paper, we consider SP scheduling as it leads to a simpler implementation and operation of the network. Focusing on SP scheduling also simplifies the analysis.

Utility Functions: The choice of utility functions significantly influences the difficulty of the analysis of price-based resource allocation schemes. As a result, a large portion of the literature focuses on utility functions which are concave. However, such utility functions do not adequately capture the service requirements of certain applications. For example, real-time voice applications require a strict minimal QoS in terms of average throughput and delay. For such an application, the utility function has a sharp transition at some QoS threshold, which leads utilities that are to monotonic (quasi-concave), but not necessarily concave. Theoretical results for quasi-concave utilities are much harder to obtain; this more challenging case has been studied by Park *et al.* in [25]. To simplify the analysis, we focus in this paper on concave utility functions.

Pricing: Pricing mechanisms for multi-class QoS provisioning can be classified into static, dynamic, or auction-based schemes. In the case of dynamic pricing, the network adapts prices as the traffic load changes. In a auction-based pricing schemes, users attach a bid to each packet indicating the willingness to pay for the delivery of the packet, and the network serves packets in descending order of their bids. Dynamic pricing is the most powerful and flexible mechanism: it can be used to achieve a socially optimal bandwidth allocation ([6], [7], [25]), maximize profit [29], and obtain an incentive-compatible class allocation [20]. Auction-based pricing mechanism have been studied for the single class case by MacKie-Mason and Varian in ([13], [14]) and for the multi-class case by Shu and Varaiya in [27]. For the multi-class case, Shu and Varaiya show that their auction-based scheme achieves an incentive-compatible class allocation. Static pricing have been studied by Jin and Kesidis in [10], and by Mandjes in [15], for the case of a single service class, and two priority classes, respectively. In [19], we analyzed static pricing for an idealized priority service with a continuum of priorities. The analysis in this paper differs with [10], [15], [19], as we consider a priority service with a general, but finite, number of priority classes. Generally, static pricing schemes are simpler to implement compared with dynamic or auction-based schemes. However, this ease of implementation comes at a cost as prices have to be chosen a priori and it is not possible to react to changes in the users demand or traffic load in the network. In this paper, we quantify "how much we loose" by using static instead of dynamic pricing. In particular, we identify the set of possible network operating points that can be achieved with static pricing.

Traffic Splitting: Multi-class QoS provisioning schemes can also be classified depending on whether users are allowed to split traffic among several classes. The situation where users can split their traffic (as considered in this paper) leads to a game theoretic setting called splittable games [25]. The case where users have to commit to single service class lead to unsplittable games [25], also called atomic games [12]. Park *et al.* analyze in [25] a multi-class QoS provisioning schemes both in the context of splittable and unsplittable games. In this paper, we allow users to split traffic among the priority classes.

Other related work is the the Paris Metro Pricing (PMP) proposal [21], [22]. PMP partitions the network into several logically separated channels where channels differ in the prices paid for using them. This scheme could be implemented with the static pricing scheme that we consider here. Cocchi *et al.* provide in [2] an experimental study which demonstrates that a price-based priority can achieve a higher a social welfare compared with the currently used best-effort/flat-rate approach. Conceptually related to our work are also network routing games where users can choose between several parallel links as considered in [23].

The rest of the paper is organized as follows. In Section II we formulate the problem that we consider. In Section III we present the link model that we use for our analysis. In Section IV we study the case where all users measure QoS solely as a function of the average throughput. We outline in Section V how this analysis can be extended to the case where users also care about packet loss. In Section VI we provide a numerical case study to illustrate our results.

II. Problem Formulation

We consider a link which is shared by R users. The link traffic is divided into N different priority classes i, $i \in \mathcal{I} = \{1, ..., N\}$, and served according to a strict priority rule, *i.e.* the link transmits priority i traffic only if all traffic with priority $j > i$ has been served. Users are free to choose the priority of their traffic, but are charged based on their traffic allocation. In particular, the link charges a price $u_i > 0$ per unit traffic submitted in priority i. Naturally, high priority traffic is more expensive and we have

$$u_i < u_j, \qquad \text{when } i < j.$$

A user accessing the link can then decide on

(a) its transmission rate, *i.e.* the amount of traffic submitted per unit time, and

(b) its traffic allocation, *i.e.* the priority of the submitted traffic.

We allow users to split their traffic among several priority classes. Suppose that $d_r(i)$, $r = 1, ..., R$, is the rate at which user r submits traffic in priority $i \in \mathcal{I}$. The aggregate rate for priority i traffic is then given by

$$d(i) = \sum_{r=1}^{R} d_r(i), \qquad i = 1, .., N.$$

Let $d_r = (d_r(1), ..., d_r(N))$ be the allocation vector of user r and let $d = (d(1), ..., d(N))$ be the aggregated allocation vector.

Consider a fixed aggregated allocation vector d and let $P_{tr}(i, d)$ be the resulting probability that priority i traffic is transmitted (and not dropped) by the link[1]. As we consider strict priority scheduling, we have that

$$P_{tr}(i, d) \le P_{tr}(j, d), \qquad \text{when } i < j.$$

Throughout the paper we assume that users are able to instantaneously determine the transmission probabilities $P_{tr}(i, d)$ for the priorities $i = 1, ..., N$. This is a simplifying assumption; in practice, users would have to probe the different priority classes to form an estimate of the transmission probabilities.

Having determined the transmission probabilities in the different priorities, users choose a traffic allocation based on their service requirements. To characterize the service requirements, we associate with each user r a utility function $U_r(x_r)$ which depends on the average throughput x_r of user r given by

$$x_r = \sum_{i=1}^{N} d_r(i) P_{tr}(i, d).$$

We make the following assumption.

Assumption 1: For each user $r = 1, ..., R$, the utility function $U_r : \Re_+ \mapsto \Re_+$, is increasing, bounded, strictly concave, and twice differentiable.

We note that utility functions with these characteristics are commonly used in the pricing literature, see for example [11]. Assumption 1 does not require that all users have the same utility function. Throughout the paper we assume that the utility function U_r is only known to user r, but unknown to all other users and the network.

Assumption 1 implies that a user can achieve a high utility by submitting high priority packets; however, doing so will also increase the cost as high priority traffic is more expensive. It is then natural to assume that each user will try to choose an allocation that strikes a balance between achieving a high utility and keeping the cost low, *i.e.* given the current aggregated allocation vector d^* and the corresponding transmission probabilities $P_{tr}(i, d^*)$, $i = 1, ..., N$, user r chooses an allocation vector d_r^* such that

$$d_r^* = \arg \max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_i d_r(i) u_i \right\}, \qquad (1)$$

where $x_r = \sum_{i=1}^{N} P_{tr}(i, d^*) d_r(i)$. Note that the term $\sum_i d_r(i) u_i$ in Eq. (1) represents the total cost that user r has to pay the link. In the above optimization problem, user r ignores how changes in its own allocation d_r impact the transmission probabilities. Indeed, if user r knows (or can accurately estimate) the aggregate allocation vector d, then it would be possible for r to incorporate the effect of its on action on the transmission probabilities. It is possible (although cumbersome) to carry out the analysis for this case. We omit here a detailed discussion and refer to [16].

As we assume that many small users share the link, changes in the allocation of an individual user has little effect on the transmission probabilities. However, the aggregate effect of all users can be substantial and cause the transmission probabilities to change significantly. As a result, users will have to constantly monitor the transmission probabilities in the different priority classes and adapt their own traffic allocation accordingly. Fluctuations in the transmission probabilities will then trigger changes (fluctuations) in the traffic allocation of the individual users, which in return will affect the transmission probabilities. It is then not clear whether this interaction among users will cause the transmission probabilities to fluctuate (oscillate) forever (leading to an unstable system and unpredictable network performance), or whether the transmission probabilities will converge and the system will reach an operating point (equilibrium). To study this situation, we define a (Wardrop) equilibrium as follows [30].

[1] We will provide the precise mathematical model for obtaining the transmission probabilities $P_{tr}(i, d)$ in Section III.

Definition 1: Let $(d_1^*, ..., d_R^*)$ be a given allocation vector such that $d_r^* \in \Re_+^N$, $r = 1, ..., R$, and let $d^* = (d^*(1), ..., d^*(N))$ be the corresponding aggregated allocation, *i.e.* we have

$$d^*(i) = \sum_{r=1}^{R} d_r^*(i), \qquad i = 1, ..., N.$$

We call $(d_1^*, ..., d_R^*)$ an equilibrium allocation when for every user $r = 1, ..., R$, we have that

$$d_r^* = \arg\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^{N} d_r(i) u_i \right\},$$

where $x_r = \sum_{r=1}^{R} d_r(i) P_{tr}(i, d^*)$.
When $(d_1^*, ..., d_R^*)$ is an equilibrium allocation, then no user r has an incentive to deviate from its allocation d_r^* as this would decrease the net benefit; hence an equilibrium is reached. The goal of this paper is to study whether such an equilibrium allocation exists, and to investigate how pricing decisions affect the link performance and bandwidth allocation, as well as the link revenue.

To characterize the bandwidth sharing properties of an equilibrium allocation, we use the following notation. For every user $r = 1, ..., R$, let the function $D_r : \Re_+ \to \Re_+$ be such that

$$D_r(u) = \arg\max_{x \geq 0} \{U_r(x) - xu\}, \quad u \in \Re_+. \qquad (2)$$

Under Assumption 1, the maximization problem given by Eq. (2) has a unique solution and $D_r(u)$ is given as follows

$$D_r(u) = \begin{cases} {U'_n}^{-1}(u) & u \leq u_{r,max}, \\ 0 & u > u_{r,max}, \end{cases}$$

where $u_{r,max} = U'_r(0)$ and $U_r'^{-1} : [0, u_{r,max}] \mapsto \Re_+$ is the inverse function of U'_r. We refer to D_r as the demand function of user r: the value $D_r(u)$ is the amount of traffic users r would submit when the network charges a price u per unit traffic and all traffic is transmitted with probability 1. We note that the function $D_r(u)$ is a well-known concept of economic theory and can be directly derived form the utility function $U_r(x_r)$; it is not tied to the specific setting considered in this paper.

III. Link Model

For our analysis, we use the following fluid-flow model. Consider a single link with a fixed capacity of C units of traffic per time slot. Given an aggregated allocation vector $d = (d(1), ..., d(N))$, traffic is served as follows. All traffic with priority i such that

$$\sum_{k=i}^{N} d(k) < C$$

is transmitted. For priority i^* with

$$\sum_{i=i^*+1}^{N} d(i) < C \quad \text{and} \quad \sum_{i=i^*}^{N} d(i) \geq C,$$

per unit time only an amount equal to $\left(C - \sum_{i=i^*+1}^{N} d(i)\right)$ - chosen at random - of the total priority i^* traffic is transmitted. All traffic with priority $i < i^*$ is dropped. The transmission probability $P_{tr}(i, d)$ for priority i traffic is then given by

$$P_{tr}(i, d) = \begin{cases} 1 & \text{if } C > \sum_{k=i}^{N} d(k), \\ \frac{C - \sum_{k=i+1}^{N} d(k)}{d(i)} & \text{if } \sum_{k=i}^{N} d(k) \geq C \\ & \text{and } C > \sum_{k=i+1}^{N} d(k), \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_{tr}(d) = (P_{tr}(1, d), ..., P_{tr}(N, d))$ be the transmission probability vector under d. The average cost for transmitting one unit of traffic in priority i is then equal to $u_i / P_{tr}(i, d)$.

IV. Analysis of User Interaction

Using the link model of the previous section, we investigate in this section whether there exists an equilibrium allocation as given by Definition 1.

We make the following two assumptions for our analysis. First, we ignore the trivial case where the link capacity exceeds the aggregate demand in the lowest priority class 1, and assume that

$$C < \sum_{r=1}^{R} D_r(u_1).$$

When this condition does not hold, then it can be shown that there exists an unique equilibrium allocation $(d_1^*, ..., d_R^*)$ which has the property that only the lowest priority 1 is used, and the allocation in priority 1 is given by

$$d_r^*(1) = D_r(u_1), \qquad r = 1, ..., R.$$

As we have that

$$\sum_{r=1}^{R} D_r(u_1) \leq C,$$

all traffic submitted in priority 1 is transmitted with probability 1, and the throughput of user r is given by

$$x_r^* = d_r^*(1) = D_r(u_1), \qquad r = 1, ..., R.$$

Second, we assume that the total demand

$$D(u_i) = \sum_{r=1}^{R} D_r(u_i), \qquad i = 1, ..., N,$$

in every priority i is strictly positive. If this is not the case for some priority i, then priority i will not be used (in equilibrium) and we can ignore it for our analysis without loss of generality. We summarize these two assumptions as follows.

Assumption 2: We have $D(u_1) > C$ and

$$D(u_i) > 0, \qquad i = 1, ..., N.$$

We start our analysis by deriving the necessary conditions for an allocation vector to be an equilibrium allocation. We use the following notation. Consider an allocation vector $(d_1^*, ..., d_R^*) \in \Re_+^R$ and let $d^* = \sum_{r=1}^R d_r^*$ be the aggregated allocation under $(d_1^*, ..., d_R^*)$. In addition, let

$$x_r^* = \sum_{i=1}^N d_r^*(i) P_{tr}(i, d^*)$$

be the average throughput of user r under $(d_1^*, ..., d_R^*)$, and let the priority i_0 be such that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i_0$$

and

$$P_{tr}(i, d^*) > 0, \text{ for all } i \geq i_0.$$

Furthermore, we define the price u_{N+1} by

$$u_{N+1} = \max_{r=1,...,R} U_r'(0),$$

where U_r' is the derivative of U_r. The price u_{N+1} is chosen such that $D(u) > 0$, for $u \in [0, u_{N+1})$ and $D(u) = 0$ for $u \geq u_{N+1}$. We then have the following result.

Proposition 1: Let Assumption 1 and 2 hold. When $(d_1^*, ..., d_R^*) \in \Re_+^R$ is an equilibrium allocation, then for all users $r = 1, ..., R$, we have that

(a) $d_r^*(i) = 0$ for $i \notin \{i_0, i_0 + 1\}$;
(b) $P_{tr}(i_0, d^*) \geq \frac{u_{i_0}}{u_{i_0+1}}$;
(c) $x_r^* = D_r(u^*)$ where

$$u^* = \frac{u_{i_0}}{P_{tr}(i_0, d^*)};$$

(d) $\sum_{r=1}^R x_r^* = C$.

A proof of Proposition 1 is given in Appendix A.

We briefly comment on Property (a)-(d) of Proposition 1. Property (a) states that only the priorities i_0 and i_0+1 are used in equilibrium. To see this, note that under the link model of Section III all traffic with a priority lower than i_0 is dropped. As the network charges users for submitted traffic, users would pay for traffic that is never delivered. Clearly, such an allocation can not be optimal. Using a priority higher than $i_0 + 1$ is not economical as this traffic is transmitted with the same probability as traffic in priority $i_0 + 1$, but at a higher cost. It can be shown that when $P_{tr}(i_0, d^*) > u_{i_0}/u_{i_0+1}$ then only priority i_0 is used and $d^*(i_0) = 0$ (see Lemma 1 in Appendix A). Property (b) states that pricing can be used to lower-bound the transmission probabilities, or equivalently, to upper-bound the loss probability. This result suggests the above pricing implements a congestion-avoidance mechanism where the loss probability of priority i traffic is upper-bounded by $1 - \frac{u_i}{u_{i+1}}$ (see discussion below). Property (c) states that in equilibrium the average transmission rate of user r is equal to the demand of user r under the price

$$u^* = \frac{u_{i_0}}{P_{tr}(i_0, d^*)}.$$

Combining Property (c) with Assumption 1 (which implies that the total demand $D(u)$ is strictly decreasing in u on $[0, u_{max}]$) and Property (d) (which states the whole link capacity is used in equilibrium), it follows the congestion price u^*, and the rate allocation vector $(x_1^*, ..., x_R^*)$, are identical under all equilibrium allocations.

In Appendix B, we show that one always can construct an equilibrium allocation and we obtain the following result.

Proposition 2: Let Assumption 1 and 2 hold. Then there exists an equilibrium allocation. The equilibrium allocation is unique when there exists a priority $i \in \mathcal{I}$ such that

$$D(u_i) > C > D(u_{i+1}).$$

It can been shown that when $D(u_i) = C$ for some priority i, then there is an infinite number of equilibrium allocations [17]. However, by Property (c) and (d) of Proposition 1, the equilibrium congestion price u^*, and the equilibrium rate allocation vector $(x_1^*, ..., x_R^*)$, are always unique.

A. Pareto Optimal Allocation

Proposition 2 establishes that there always exists an equilibrium allocation. However, we would like to point out that an equilibrium allocation is (generally) not Pareto optimal with respect to the net benefit. A Pareto optimal allocation would have the property that there exists no other allocation that achieves a higher net benefit for one of the users without decreasing the net benefit of any other user. Indeed, it is not difficult to see that each user would (generally) do better (and no user would do worse) when all users keep submitting the same amount of traffic as under an equilibrium, but use the lowest priority 1 instead of the priorities i_0 and $i_0 + 1$. This result is not surprising as it is well known that when users behave in a selfish manner and are only interested in maximizing their own net benefit, then the resulting equilibrium is not necessarily Pareto optimal.

B. Bandwidth Allocation

Proposition 1 characterizes how the link bandwidth is shared among users in equilibrium. We obtain the following result. In equilibrium, the average throughput x_r^* of user r is given by

$$x_r^* = D_r(u^*), \qquad r = 1, ..., R,$$

where u^* is the unique price such that

$$\sum_{r=1}^R x_r^* = \sum_{r=1}^R D_r(u^*) = C.$$

We note that rate allocation is the same as obtained under the smart market [13], [14]. The price u^* is the clearance price for which the total demand is equal to the link capacity. Note that the price u^* depends solely on the aggregated demand function. Somewhat surprising, this results states that the equilibrium bandwidth allocation does not depend on the prices u_i, $i = 1, ..., N$.

C. Revenue Maximization

The above results implies the link revenue is also independent of the prices u_i, $i = 1, ..., N$.

Corollary 1: Let Assumption 1 and 2 hold. In equilibrium, the revenue obtained per unit time is equal to $u^* D(u^*)$, where u^* is the price of Proposition 1.

Proof: Suppose that there exists a priority i_0 such that $D(u_{i_0}) > C > D(u_{i_0+1})$. In this case, the equilibrium allocation is unique and has the property that $d^*(i) = 0$ for $i \neq i_0$. The revenue per unit time is then equal to

$$d(i_0)u_{i_0} = d(i_0)P_{tr}(i_0, d^*)u^* = d(i_0)\frac{C}{d(i_0)}u^*,$$

where we used that $u^* = u_{i_0}/P_{tr}(i_0, d^*)$. In equilibrium, we have by Proposition 1 that $C = D(u^*)$ and it follows that

$$d(i_0)u_{i_0} = u^* D(u^*).$$

The above argument can also be applied to the case where the equilibrium is not unique. We omit a detailed derivation for this case. ■

D. Congestion Control

Property (b) of Proposition 1 states that when priority i is used (in equilibrium), then the transmission probability for priority i traffic is lower-bounded (in equilibrium) by the price ratio u_i/u_{i+1}. This result has the following interpretation. Note that the average cost for transmitting traffic in priority i under the allocation $(d_1^*, ..., d_R^*)$ is equal to $u_i/P_{tr}(i, d^*)$. When the transmission probability $P_{tr}(i, d^*)$ falls below the threshold u_i/u_{i+1}, then it is (on average) more expensive to transmit traffic in priority i than in priority $i + 1$ (where traffic is transmitted with probability 1 at cost u_{i+1}). Therefore, whenever priority i becomes congested, all users will switch their traffic from priority i to $i + 1$. In addition, as priority $i + 1$ traffic is more expensive, users will also reduce their transmission rates. This implies that the above price-based allocation scheme provides a congestion-avoidance mechanism similar to the congestion-avoidance mechanism of TCP. However, the above scheme uses pricing to ensure that users reduce their transmission rates when packet loss increases (and the transmission probabilities decrease), whereas TCP assumes that users voluntarily reduce their transmission rate when packet loss occurs.

E. Use of Priority Classes

Property (a) of Proposition 1 states that in equilibrium at most two priority classes are used. However, note that it is not known a priori which two priorities are potentially used as this is a function of the exogenous demand. As the user populations and demand changes, the used priority classes will also change. Note that this result illustrates that the use of several priority classes with different prices hedges against uncertainty in the exogenous demand and a minimal level of QoS in terms of packet loss can be guaranteed, as long as price u_N for the highest priority is chosen large enough. More precisely, suppose that $R(t)$ is the number of customers accessing the link at time $t \geq 0$ and $D_r(u, t)$, $r = 1, ..., R(t)$, are the corresponding demand functions. To ensure that the transmission probability in any priority class never falls below a required value P_0, the number of priority classes N and prices $u_1, ..., u_N$ should be chosen such that

$$\max_{i \in \{1,...,N-1\}} \frac{u_i}{u_{i+1}} \geq P_0$$

and

$$\sum_{r=1}^{R(t)} D_r(u_N, t) \leq \frac{C}{P_0}, \quad \text{for all } t \geq 0.$$

Note that this procedure still requires some limited knowledge about the exogenous demand. However, rather than knowing the exact shape of the demand function, we only have to know that the demand under the price u_N never (or very rarely) exceeds the threshold C/P_0.

F. Role of Pricing

The above results shed some light on the role of pricing in static pricing schemes for priority services. It suggests that prices do not influence the equilibrium bandwidth allocation and the link revenue. However, prices do affect the QoS obtained in the different priority classes and static pricing can be used to guarantee a minimal QoS in the different priority classes in terms of packets loss.

V. Packet Loss

In this section, we indicate how the above analysis can be extended to the case where users have strict QoS requirements regarding packet loss.

For users with strict QoS requirements regarding packet-loss, the utility obtained from accessing the link decreases when (parts of) its traffic is dropped by the link; or in other words, the utility of a user r decreases when user r submits traffic in a priority i for which the transmission probability $P_{tr}(i, d)$ is strictly less than 1. To model this situation, we add to the utility of user r a (penalty) term c_r which decreases the utility when packet-loss occurs. The (total) utility of user r is then given by

$$U_r(x_r) - \sum_{i=1}^{N} c_r\Big(P_{tr}(i, d)\Big) d_r(i), \quad r = 1, .., R. \qquad (3)$$

We make the following assumption.

Assumption 3: For each user $r = 1, ..., R$,, the function $c_r : [0, 1] \rightarrow [0, \infty)$ is bounded, non-increasing, and continuous, with $c_r(1) = 0$.

For users with loss sensitive traffic, we let the function $c_r(P_{tr}(i, d))$ increase (and the utility decrease) as the transmission probability $P_{tr}(i, d)$ falls below 1. The slope of c_r determines how sensitive user r is towards packet loss. For users with loss-insensitive traffic, c_r is equal to 0 on $[0, 1]$.

In this setting, an allocation $(d_1^*, ..., d_R^*)$ is an equilibrium allocation when for every user $r = 1, ..., R$, we have that

$$d_r^* = \arg \max_{d_r \in \Re_+^N} \Big\{ U_r(x_r) \quad - \quad \sum_{i=1}^{N} c_r\Big(P_{tr}(i, d)\Big) d_r(i)$$

$$- \sum_{i=1}^{N} d_r(i)u_i\Big\},$$

where $x_r = \sum_{i=1}^{N} d_r(i)P_{tr}(i,d^*)$. We have the following result.

Proposition 3: Let Assumption 1, 2, and 3 hold. When $(d_1^*,...,d_R^*)$ is an equilibrium allocation, then for all users $r=1,...,R$, we have that

(a) $d_r^*(i) = 0$ for $i \notin \{i_0, i_0+1\}$;
(b) $P_{tr}(i_0,d^*) \geq \frac{u_{i_0}}{u_{i_0+1}}$;
(c) $x_r^* = D_r(u_r^*)$, for

$$u_r^* = \min\left\{u_{i_0+1}, \frac{u_{i_0} + c_r\Big(P_{tr}(i_0,d^*)\Big)}{P_{tr}(i_0,d^*)}\right\};$$

(d) $\sum_r x_r^* = C$.

We provide a proof for Proposition 3 in Appendix C.

Proposition 3 is identical to Proposition 1, except that the price u_r^* (which determines the average throughput of user r) now also depends on the QoS requirements with respect to packet loss (see Property (c)). For a user r with loss-insensitive traffic, the price u_r^* is equal to $u_{i_0}/P_{tr}(i_0,d^*)$ as $c_r(P_{tr}(i_0,d^*))$ is equal to 0 on $[0,1]$. For a user r with loss-sensitive traffic, the price u_r^* is equal to u_{i_0+1} when the term $c_r\Big(P_{tr}(i_0,d^*)\Big)$ is large and

$$u_{i_0+1} < \frac{u_{i_0} + c_r\Big(P_{tr}(i_0,d^*)\Big)}{P_{tr}(i_0,d^*)}.$$

From Property (b), we have that

$$u_{i_0+1} \geq \frac{u_{i_0}}{P_{tr}(i_0,d^*)},$$

and it follows that users with loss-sensitive traffic experience a higher price u_r^* than users with loss-insensitive traffic. Moreover, one can show that users with loss-insensitive traffic will allocate their traffic to priority i_0, whereas users with loss-sensitive traffic will submit their traffic in priority i_0+1 to obtain a better QoS regarding with respect to packet loss. This result shows that this price-based allocation scheme provides differentiated QoS, and allows users with tight QoS requirements to obtain a better QoS by paying a higher price u_r^*.

Again, it can be shown that there always exists an equilibrium allocation for the case where users with tight QoS requirements access the link. However, we were not able to find (simple) conditions for the existence of a unique equilibrium, as this was the case in Proposition 2 of the previous section.

Proposition 4: Let Assumption 1, 2, and 3 hold. Then there exists an equilibrium allocation.

We provide a proof for Proposition 4 in Appendix C.

VI. Dynamic Traffic Allocation

In this section, we consider the situation where users dynamically update their allocation, and investigate whether the resulting traffic allocations converge to an equilibrium. For our analysis, we use an adaptation model where users first form an estimate of the transmission probabilities in the different priorities, and then choose their allocation by solving the maximization problem given by Eq. 1.

Suppose that each user starts with an arbitrary initial allocation and then updates its allocation at discrete time steps to maximize its net benefit. Let $d^{(k)} = (d^{(k)}(1),...,d^{(k)}(N))$ be the aggregate allocation at time k, for $k = 0,1,2,...$, and let $P_{tr}(i,d^{(k)})$, $i=1,...,N$, be the resulting transmission probabilities. In addition, suppose that at the beginning of time step k all users know the transmission probabilities of the previous time step $k-1$. We then consider the following adaption model. At time step k, user r first forms an estimate $\hat{P}_{tr,r}^{(k)} = \Big(\hat{P}_{tr,r}^{(k)}(1),...,\hat{P}_{tr,r}^{(k)}(N)\Big)$ for the transmission probabilities at time step k by setting

$$\hat{P}_{tr,r}^{(k)}(i) = \hat{P}_{tr,r}^{(k-1)}(i) + \alpha\Big(P_{tr}(i,d^{(k-1)}) - \hat{P}_{tr,r}^{(k-1)}(i)\Big), \quad (4)$$

where $\hat{P}_{tr,r}^{(k-1)}(i)$ is the estimate at time $k-1$, $\alpha > 0$ is a small step size parameter, and $(P_{tr}(i,d^{(k-1)}) - \hat{P}_{tr,r}(i))$ is the estimation error of the transmission probability for priority i at time $k-1$.

At time k, user r then chooses an allocation $d_r^{(k)}$ based on the estimate $\Big(\hat{P}_{tr,r}^{(k)}(1),...,\hat{P}_{tr,r}^{(k)}(N)\Big)$ and sets

$$d_r^{(k)} = \arg\max_{d_r \in \Re^N}\left\{U_r(x_r) - \sum_i d_r(i)u_i\right\},$$

where $x_r = \sum_{i=1}^{N} \hat{P}_{tr,r}^{(k)}(i)d_r(i)$.

A similar adaptation model has been proposed, and analyzed, by Ganesh and Laevens [4]. A detailed analysis of the above adaptation algorithm is beyond the scope of this paper and will be presented in a forthcoming paper. We use a numerical case study to illustrate its convergence properties.

We consider a single communication link with capacity $C=1$. The link supports $N=5$ priority levels and the prices associated with the different priorities are given by $u_i = 2^{(i-1)}$, $i=1,...,5$. There are $R=50$ users accessing the link. We choose piece-wise linear functions to represent the users' demand and set

$$D_r(u) = \max\{0, 1.6 - a_r u\}, \qquad r=1,...,R,$$

where $a_r = 0.1\,(1-1/2r)$.

As initial parameters, we set $\hat{P}_{tr,r}^{(0)}(i) = 1$ and

$$d_r(i)^{(0)} = \begin{cases} D_r(u_i) & i=1, \\ 0 & \text{otherwise}, \end{cases}$$

for all users r and all priorities i. The step size parameter α in Eq. (4) is equal to 0.05. We then simulated the above adaptation model for 100 time steps. The resulting trajectories of the transmission rates $x_r^{(k)}$,

$$x_r^{(k)} = \sum_{i=1}^{N} P_{tr}^{(k)}(i,d^{(k)})d_r(i), \qquad r=1,...,R,$$

are shown in Figure 1. The trajectory of the aggregated demand $d^{(k)}(i) = \sum_{r=1}^{R} d^{(k)}(i)$ in the priorities $i = 1, ..., N$ is given by Figure 2. Figure 3 depicts the trajectory of transmission probabilities $P_{tr}(d^{(k)}, i)$, $i = 1, ..., 5$, and Figure 4 shows the trajectory of the estimated transmission probabilities $\hat{P}_{tr}(d^{(k)}, i)$, $i = 1, ..., 5$. As all users have the same initial estimates, the estimates of the transmission probabilities are identical at every time step.

We make the following observations. All users reach within 50 steps the equilibrium transmission rate (see Figure 1). In equilibrium, only priority $i^* = 3$ is used (see Figure 2), and the transmission probability for priority 3 is equal 0.69 in equilibrium (see Figure 3). As predicted by Proposition 1, this transmission probability is lower-bounded by the price ratio $u_3/u_4 = 0.5$. Between time step $k = 0$ and $k = 15$, only priority 1 is used (see Figure 2). At time $k = 15$, the estimate of the transmission probability in priority 1 drops below the lower bound $u_1/u_2 = 0.5$ (see Figure 4) and all users switch their traffic to priority 2 (see Figure 2). Similarly, at time $k = 30$, the estimated transmission probability in priority 2 drops below $u_2/u_3 = 0.5$ and all users switch their traffic to priority 3. In addition, users reduce the total amount of traffic submitted to the link when they switch traffic to the next higher priority (see Figure 2). This illustrates the discussion of Subsection IV-D where we observed that the above pricing scheme implements a congestion-avoidance mechanism, where users adapt their transmission rates based on feedback information on the transmission probabilities in the different priority classes.

VII. Conclusions

We analyzed a static pricing scheme for priority services in a game-theoretic framework. For the resulting game, we showed that there always exists an equilibrium allocation. Furthermore, we showed that the equilibrium allocation is unique, except for the (rare) case when there exists a priority i such that the total demand $D(u_i)$ under the price u_i is equal to the link capacity. We characterized the bandwidth allocation and link revenue in equilibrium and showed that they do not depend on the prices of the different priorities. Our analysis also revealed a simple relation between pricing decisions and the packet loss probabilities. In particular, when a packet with priority i is submitted under an equilibrium allocation $(d_1^*, ..., d_R^*)$, then the packet transmission probability $P_{tr}(i, d^*)$ is lower bounded by u_i/u_{i+1}. For the case where users with multidimensional QoS requirements access the link, we showed that the above priority service provides differentiated QoS: users with tight QoS requirements are willing to pay a higher price to use high priorities and obtain a better QoS.

The analysis presented in this paper can be extended to bursty traffic sources [18]. For future work, it would be interesting to extend the analysis of this paper to the network case. Such an extension seems possible, but poses additional technical challenges. In particular, necessary conditions for an equilibrium allocation (as given by Proposition 1) can easily be extended to the network case, how-

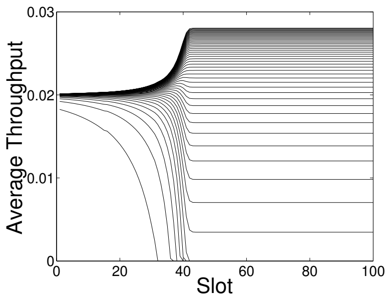

Fig. 1. Trajectories of the average throughput x_r, $r = 1, ..., R$.

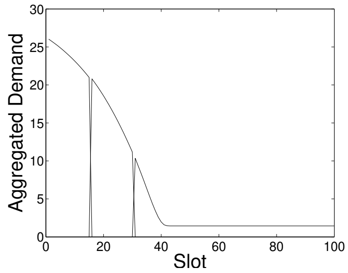

Fig. 2. Trajectories of the aggregated demand $d^{(k)}(i)$ in priority $i = 1, ..., 5$. Between time step $k = 0$ and $k = 15$, only priority 1 is used, and the demand in all other priorities is equal to 0. At time $k = 15$, all users switch their traffic from priority 1 to priority 2, and at time $k = 30$ all users switch traffic from priority 2 to priority 3. In equilibrium, only priority 3 is used.

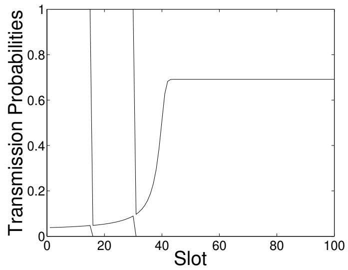

Fig. 3. Trajectories of the transmission probabilities $P_{tr}(i, d^{(k)})$ for $i = 1, ..., 5$. The bottom curve is the trajectory for the transmission probability in the lowest priority 1; the top curve is the trajectory for priority 3. The transmission probability in priority 4 and 5 is always equal to 1.

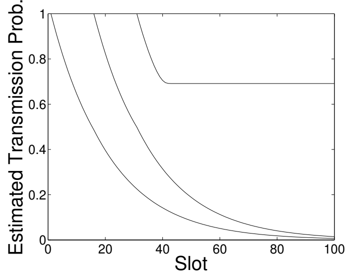

Fig. 4. Trajectories of the estimated transmission probabilities $\hat{P}_{tr}(i, d^{(k)})$ for $i = 1, ..., 5$. The bottom curve is the trajectory for the transmission probability in the lowest priority 1; the top curve is the trajectory for priority 3. The estimated transmission probability in priority 4 and 5 is always equal to 1.

ever showing the existence of an equilibrium (as given by Proposition 2) is more involved.

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Appendix

I. Proof of Proposition 1

We first consider the case where only one user chooses an allocation to maximize its net benefit, and assume that the allocations of all other users are fixed. We use the following notation. Let $d^* \in \Re_+^N$ be a given aggregate allocation and let the priority $i^* \in \{1, ..., N\}$ be such that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i^*,$$

and

$$P_{tr}(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Let $d_r^* \in \Re_+^N$ be the allocation of a given user $r \in \{1, ..., R\}$, and let $x_r^* = \sum_{i=1}^N P_{tr}(i, d^*) d_r^*(i)$ be the average throughput of user r under d_r^*. Furthermore, we define $u_{r,\max}$ by $u_{r,max} = U_r'(0)$ where U_r' is the derivative of U_r. Note that the price $u_{r,max}$ is such that $D_r(u) > 0$, for $u \in [0, u_{r,max})$ and $D_r(u) = 0$ for $u \geq u_{r,max}$.

Lemma 1: Let Assumption 1 and 2 hold. Then $d_r^* \in \Re_+^N$ is an optimal solution for the maximization problem

$$\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^N d_r(i) u_i \right\},$$

where

$$x_r = \sum_{i=1}^N P_{tr}(i, d^*) d_r(i),$$

if and only if the following holds.

(1) For $i \notin \{i^*, i^*+1\}$, we have $d_r^*(i) = 0$

(2) When $d_r^*(i^*) > 0$, then $U_r'(x_r^*) = \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$ and $P_{tr}(i^*, d^*) \geq \frac{u_{i^*}}{u_{i^*+1}}$.

(3) When $d_r^*(i^*+1) > 0$, then $U_r'(x_r^*) = u_{i^*+1}$ and $P_{tr}(i^*, d^*) \leq \frac{u_{i^*}}{u_{i^*+1}}$.

(4) When $d_r(i^*) = d_r(i^*+1) = 0$, then $u_{r,\max} \leq \frac{u_{i^*}}{P_{tr}(i^*,d^*)}$ and $u_{r,\max} \leq u_{i^*+1}$.

Furthermore, when $d_r^* \in \Re_+^N$ is an optimal solution for the above maximization problem such that $P_{tr}(i^*,d^*) > \frac{u_{i^*}}{u_{i^*+1}}$, then we have that $d_r(i^*+1) = 0$.

Proof: We will show that Properties (1)-(4) are necessary and sufficient conditions for d_r^* to be an optimal solution for the maximization problem given in the proposition.

We use the following notation. Let $d_r^* \in \Re_+^N$ be as given in the statement of the lemma, and for $d_r \in \Re_+^N$ let the function $F_r(d_r,d^*)$ be given by

$$F_r(d_r,d^*) = U_r(x_r) - \sum_{i=1}^{N} d_r(i)u_i,$$

where $x_r = \sum_{i=1}^{N} P_{tr}(i,d^*)d_r(i)$. Furthermore, for $r = 1,...,R$, let $f_{r,i}(d_r,d^*)$ be the derivative of $F_r(d_r,d^*)$ with respect to $d_r(i)$ for $i = 1,...,N$, *i.e.* we have

$$f_{r,i}(d_r,d^*) = \lim_{\alpha \to 0} \frac{F_r(d_r + \alpha e_i, d^*) - F_r(d_r,d^*)}{\alpha}$$

where e_i is the ith unit vector. These derivatives exist as, by Assumption 1, $U_r(x_r)$ is differentiable on $[0,\infty)$. In particular, we have

$$f_{r,i}(d_r,d^*) = U_r'(x_r)P_{tr}(i,d^*) - u_i,$$

where U_r' is the derivative of U_r (with respect to x_r). The first order conditions for $d_r \in \Re_+^N$ to be an optimal solution for the maximization problem given in the statement of the lemma are then given as follows,

$$f_{r,i}(d_r,d^*) \leq 0, \qquad \text{for } i = 1,...,N, \tag{5}$$

where this equation holds with equality when $d_r(i) > 0$. In the following, we show these first order conditions hold, if and only if the Properties (1)-(4) of Lemma 1 hold. Furthermore, note that by Assumption 1, we have that the utility function U_r is concave in d_r and therefore the first order conditions are sufficient conditions.

Let the aggregate allocation $d^* \in \Re_+^N$ and the priority i^* be as given in the statement of the Lemma, *i.e.* we have that

$$P_{tr}(i,d^*) = 0, \text{ for all } i < i^*,$$

and

$$P_{tr}(i,d^*) > 0, \text{ for all } i \geq i^*.$$

Note that, for $i < i^*$ we have

$$f_{r,i}(d_r,d^*) = -u_i < 0, \qquad d_r \in \Re_+^N.$$

Furthermore, for $i > i^*+1$ and $d_r \in \Re_+^N$, we have

$$f_{r,i}(d_r,d^*) = U_r'(x_r) - u_i < U_r'(x_r) - u_{i^*+1} = f_{r,i^*+1}(d_r,d^*).$$

This implies the following. When

$$f_{r,i}(d_r,d^*) \leq 0, \qquad \text{for } i = i^*, i^*+1,$$

then we have

$$f_{r,i}(d_r,d^*) < 0, \qquad \text{for } i \notin \{i^*, i^*+1\}.$$

Combining this result with the first order conditions (see Equation (5)), we obtain that when $d_r^* \in \Re_+^N$ is an optimal solution, then we have

$$d_r^*(i) = 0, \qquad \text{for } i \notin \{i^*, i^*+1\}.$$

This establishes Property (1).

To satisfy Equation (5) for $i = i^*$ and $i = i^*+1$, we need the following. When $d_r(i^*) > 0$, then we have that

$$f_{r,i^*}(d_r,d^*) = U_r'(x_r)P_{tr}(i^*,d^*) - u_{i^*} = 0$$

and

$$f_{r,i^*+1}(d_r,d^*) = U_r'(x_r) - u_{i^*+1} \leq 0.$$

These two equations imply that

$$U_r'(x_r) = \frac{u_{i^*}}{P_{tr}(i^*,d^*)} \leq u_{i^*+1}$$

and

$$P_{tr}(i^*,d^*) \geq \frac{u_{i^*}}{u_{i^*+1}}.$$

This gives Property (2). Similarly, when $d_r(i^*+1) > 0$, then we have that

$$U_r'(x_r) = u_{i^*+1} \leq \frac{u_{i^*}}{P_{tr}(i^*,d^*)}$$

and

$$P_{tr}(i^*,d^*) \leq \frac{u_{i^*}}{u_{i^*+1}},$$

which gives Property (3).

Finally, when $d_r^* \in \Re_+^N$ is an optimal solution with $d_r(i^*) = d_r(i^*+1) = 0$, then Property (1) implies that $x_r = 0$. Therefore, we need the following to satisfy Equation (5),

$$f_{r,i^*}(d_r,d^*) = U_r'(0)P_{tr}(i^*,d^*) - u_{i^*} \leq 0$$

and

$$f_{r,i^*+1}(d_r,d^*) = U_r'(0) - u_{i^*+1} \leq 0.$$

Combining these two equations, we obtain that $u_{r,\max} = U_r'(0) \leq \frac{u_{i^*}}{P_{tr}(i^*,d^*)}$ and $u_{r,\max} \leq u_{i^*+1}$. This gives Property (4).

So far, we have shown that when the first order conditions given by Equation (5) are satisfied, then the Properties (1)-(4) hold. Similarly, one can show that when the Properties (1)-(4) hold, then the first order conditions are satisfied. We omit a detailed derivation here.

To show that Property (5) holds, we proceed as follows. Assume that $P_{tr}(i^*,d^*) > \frac{u_{i^*}}{u_{i^*+1}}$. Then we have

$$\begin{aligned} f_{r,i^*}(d_r,d^*) &= U_r'(x_r)P_{tr}(i^*,d^*) - u_{i^*} \\ &> U_r'(x_r) - u_{i^*+1} \\ &= f_{r,i^*+1}(d_r,d^*), \end{aligned}$$

and the first order conditions given by Equation (5) imply that $d_r(i^*+1) = 0$. ■

We use Lemma 1 to prove Proposition 1.

Proof of Proposition 1:

Property (a): This property follows from Property (1) of Lemma 1.

Property (b): Note that when $P_{tr}(i^*, d^*) < 1$, then there exists a user $r \in \{1, ..., R\}$ such that $d_r^*(i^*) > 0$. Property (b) then follows from Property (2) of Lemma 1.

Property (c): Consider a fixed user $r \in \{1, ..., R\}$. When $d_r^*(i^*) > 0$, then Property (c) follows from Property (2) of Lemma 1. When $d_r^*(i^*+1) > 0$, then by combining Property (3) of Lemma 1 with Property (b), we obtain that

$$P_{tr}(i^*, d^*) = \frac{u_{i^*}}{u_{i^*+1}}$$

and

$$U_r'(x_r^*) = u_{i^*+1} = \frac{u_{i^*}}{P_{tr}(i^*, d^*)}.$$

This implies Property (c). Finally, when $d_r^*(i^*) = d_r^*(i^*+1) = 0$, then from Property (a) we have that $x_r^* = 0$, and from Property (4) of Lemma 1 we have that

$$u_{r,\max} \le \frac{u_{i^*}}{P_{tr}(i^*, d^*)}.$$

As $D_r(u) = 0$ for all $u \ge u_{r,\max}$, it follows that

$$x_r^* = D_r\left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right).$$

Property (d): Note that

$$\begin{aligned} \sum_{r=1}^{R} x_r^* &= \sum_{r=1}^{R} P_{tr}(i^*, d^*) d_r(i^*) + \sum_{r=1}^{R} d_r^*(i^*+1) \\ &= P_{tr}(i^*, d^*) \sum_{r=1}^{R} d_r(i^*) + \sum_{r=1}^{R} d_r^*(i^*+1) \\ &= \frac{C - \sum_{r=1}^{R} d_r^*(i^*+1)}{\sum_{r=1}^{R} d_r^*(i^*)} \sum_{r=1}^{R} d_r^*(i^*) \\ &\quad + \sum_{r=1}^{R} d_r^*(i^*+1) \\ &= C. \end{aligned}$$

■

II. Proof of Proposition 2

In the next lemma, we derive sufficient conditions for determining whether a vector $(d_1^*, ..., d_R^*)$ is an equilibrium allocation.

Lemma 2: Let Assumption 1 and 2 hold. Then the allocation $(d_1^*, ..., d_R^*)$ is an equilibrium allocation when the following properties hold for all users $r = 1, ..., R$.

(a) $d_r^*(i) = 0$, for $i \neq i^*$.

(b) $d_r^*(i^*) = \frac{1}{P_{tr}(i^*, d^*)} D_r\left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right)$.

(c) $P_{tr}(i^*, d^*) \ge \frac{u_{i^*}}{u_{i^*+1}}$.

(d) $U_r'(x_r^*) P_{tr}(i^*, d^*) - u_{i^*} \le 0$, where we have equality when $d_r(i^*) > 0$.

Proof: When the Properties (a)-(d) hold, then Lemma 1 applies for each allocation d_r^*, $r = 1, ..., R$, and $(d_1^*, ..., d_R^*)$ is an equilibrium allocation. In particular, Property (1) of Lemma 1 follows from Property (a) of Lemma 2. Furthermore, when $d_r^*(i^*) > 0$, then by Property (b) of Lemma 2 we have

$$x_r^* = P_{tr}(i^*, d^*) d_r^*(i^*, d^*) = D_r\left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right).$$

It then follows that

$$U_r'(x_r^*) P_{tr}(i^*, d^*) - u_{i^*} = \left(\frac{u_{i^*}}{P_{tr}(i^*, d^*)}\right) P_{tr}(i^*, d^*) - u_{i^*} = 0,$$

and Property (2) of Lemma 1 holds for user r. When $d_r^*(i^*) = 0$, then by Property (b) of Lemma 2 we have that $u_{r,\max} \le \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$. It follows that

$$\begin{aligned} U_r'(x_r^*) P_{tr}(i^*, d^*) - u_{i^*} &= U_r'(0) P_{tr}(i^*, d^*) - u_{i^*} \\ &= u_{r,\max} P_{tr}(i^*, d^*) - u_{i^*} \\ &\le 0, \end{aligned}$$

and Property (4) of Lemma 1 holds for user r. ■

We use Lemma 2 to construct an equilibrium allocation and prove Proposition 2.

Proof of Proposition 2: Let i_0 be the priority for which

$$D(u_{i_0}) > C, \text{ and } D(u_{i_0+1}) \le C.$$

Such a priority i_0 exists as, by Assumption 1 and 2, we have that $D(u_1) > C$ and $D(u_{N+1}) = 0$. For $\rho \in (0, 1]$, consider the allocations $d_r(\rho)$, $r = 1, ..., R$, given by

$$d_r(i, \rho) = \begin{cases} \frac{1}{\rho} D_r\left(\frac{u_{i_0}}{\rho}\right), & \text{for } i = i_0 \\ 0, & \text{otherwise} \end{cases}$$

For $\rho = 1$, we obtain that

$$\frac{C}{\sum_{r=1}^{R} d_r(i_0, 1)} = \frac{C}{D(u_{i_0})} < 1.$$

Similarly, for $\rho = \frac{u_{i_0}}{u_{i_0+1}}$, we have

$$\frac{C}{\sum_{r=1}^{R} d_r(i_0, \rho)} = \frac{C}{D(u_{i_0+1})} \frac{u_{i_0}}{u_{i_0+1}} \ge \frac{u_{i_0}}{u_{i_0+1}}.$$

Note that $d_r(\rho)$ is continuous on $(0, 1]$, and it follows that there exists a $\rho^* \in [u_{i_0}/u_{i_0+1}, 1]$ such that

$$\frac{C}{\sum_{r=1}^{R} d_r(i_0, \rho^*)} = \rho^*, \quad r = 1, ..., R.$$

By construction, the allocation vector $(d_1(\rho^*)), ..., d_R(\rho^*))$ satisfies all properties of Lemma 2. This establishes that there exists an equilibrium allocation.

Next, we assume that there exists a priority $i_0 \in \{1, ..., N\}$ be such that $D(u_{i_0+1}) < C < D(u_{i_0})$, and show that there exists a unique equilibrium. Consider an allocation vector $(d_1^*, ..., d_R^*)$ and let $d^* = \sum_{r=1}^R d_r^*$. Furthermore, let the priority $i^* \in \{1, ..., N\}$ be such that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i^*,$$

and

$$P_{tr}(i, d^*) > 0, \text{ for all } i \geq i^*.$$

When $i^* > i_0$, then $(d_1^*, ..., d_R^*)$ is not an equilibrium allocation. To see this, note that in this case we have

$$D(u) < C, \qquad \text{for all } u \in [u_{i^*}, u_{i^*+1}],$$

and either Property (b), or Property (d), of Proposition 1 will not hold. Similarly, when $i^* < i_0$, then $(d_1^*, ..., d_R^*)$ is not an equilibrium allocation as we have

$$D(u) > C, \qquad \text{for all } u \in [u_{i^*}, u_{i^*+1}].$$

Therefore, it suffices to consider the case where $i^* = i_0$.

For this case, assume that $(d_1^*, ..., d_R^*)$ is an equilibrium allocation, and let u^* be such that $D(u^*) = C$. Note that by assumption we have that $D(u_{i_0}) > C > D(u_{i_0+1})$, and it follows that

$$u_{i^*} < u^* < u_{i^*+1}.$$

Properties (b)-(d) of Proposition 1, together with Property (5) of Lemma 1, then imply that

$$P_{tr}(i^*, d^*) > \frac{u_{i^*}}{u_{i^*+1}}$$

and

$$d_r^*(i^* + 1) = 0, \qquad \text{for } r = 1, ..., R.$$

Combining the above equation with Property (a) of Proposition 1, it follows that when $(d_1^*, ..., d_R^*)$ is an equilibrium allocation, then we have

$$d_r^*(i) = 0, \quad \text{for } i \neq i^* \text{ and } r = 1, ..., R. \tag{6}$$

Combining Equation 6 with Property (b) and (c) of Proposition 1, we obtain that

$$x_r^* = \sum_{i=1}^N P_{tr}(i, d^*) d_r^*(i) = P_{tr}(i^*, d^*) d_r^*(i^*) = D_r(u^*),$$

and

$$\frac{u_{i^*}}{P_{tr}(i^*, d^*)} = u^*.$$

This implies that

$$d_r^*(i^*) = \frac{u^*}{u_{i^*}} D_r(u^*), \qquad \text{for } r = 1, ..., R.$$

Furthermore, from Assumption 1 (which implies that the total demand $D(u)$ is strictly decreasing on $[0, u_{N+1}]$) it follows that the price u^* is uniquely given by the capacity C and the function $D(u)$; and we have that the allocation vector $(d_1^*, ..., d_R^*)$ is unique, *i.e.* we have that for each user $r = 1, ..., R$ the allocation d_r^* is uniquely given by

$$d_r^*(i^*) = \frac{u^*}{u_{i^*}} D_r(u^*).$$

This establishes that existence of a unique equilibrium allocation for the case where there exists a priority $i_0 \in \{1, ..., N\}$ be such that

$$D(u_{i_0+1}) < C < D(u_{i_0}).$$

■

III. Proof of Proposition 3 and 4

To prove Proposition 3 and 4, we proceed similarly as for Proposition 1 and 2. Again, we first consider the case where only one user chooses an allocation to maximize its net benefit, and assume that the allocations of all other users are fixed.

Let $d^* \in \Re_+^N$ be a given aggregate allocation and let the priority $i^* \in \{1, ..., N\}$ be such that

$$P_{tr}(i, d^*) = 0, \text{ for all } i < i^*,$$

and

$$P_{tr}(i, d^*) > 0, \text{ for all } i \geq i^*.$$

Let $d_r^* \in \Re_+^N$ be the allocation of a given user $r \in \{1, ..., R\}$, and let $x_r^* = \sum_{i=1}^N P_{tr}(i, d^*) d_r^*(i)$ be the average throughput of user r under d_r^*. Furthermore, we define $u_{r,\max}$ by $u_{r,max} = U_r'(0)$, where U_r' is the derivative of U_r.

Lemma 3: Let Assumption 1, 2 and 3, hold. Then $d_r^* \in \Re_+^N$ is an optimal solution to the maximization problem

$$\max_{d_r \in \Re_+^N} \left\{ U_r(x_r) - \sum_{i=1}^N c_r\Big(P_{tr}(i, d^*)\Big) d_r(i) - \sum_{i=1}^N d_r(i) u_i \right\},$$

where

$$x_r = \sum_{i=1}^N P_{tr}(i, d^*) d_r(i),$$

if and only if the following holds.

(1) $d_r(i) = 0$, for $i \notin \{i^*, i^* + 1\}$.

(2) When $d_r^*(i^*) > 0$, then $U_r'(x_r^*) = \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{P_{tr}(i^*, d^*)}$ and $P_{tr}(i^*, d^*) \geq \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{u_{i^*+1}}$.

(3) When $d_r^*(i^* + 1) > 0$, then $U_r'(x_r^*) = u_{i^*+1}$ and $P_{tr}(i^*, d^*) \leq \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{u_{i^*+1}}$.

(4) When $d_r(i^*) = d_r(i^* + 1) = 0$, then $u_{\max,r} \leq \frac{u_{i^*}}{P_{tr}(i^*, d^*)}$ and $u_{\max,r} \leq u_{i^*+1}$.

(5) When $P_{tr}(i^*, d^*) > \frac{u_{i^*} + c_r(P_{tr}(i^*, d^*))}{u_{i^*+1}}$, then $d_r(i^* + 1) = 0$.

Proof: This lemma can be proven using the same argument as given in Lemma 1. In particular, for $d_r \in \Re_+^N$, the function $F_r(d_r, d^*)$ is now given by

$$F_r(d_r, d^*) = U_r(x_r) - \sum_{i=1}^N d_r(i) u_i - \sum_{i=1}^N d_r(i) c_r\Big(P_{tr}(i, d^*)\Big),$$

where $x_r = \sum_{i=1}^N P_{tr}(i, d^*) d_r(i)$. The derivatives $f_{r,i}(d_r, d^*)$ of $F_r(d_r, d^*)$ with respect to $d_r(i) \in \Re_+$ are then given by

$$f_{r,i}(d_r,d^*) = \begin{cases} -u_i - c_r(0), & \text{for } i < i^* \\ U_r'(x_r)P_{tr}(i^*,d^*) - u_{i^*} - ... \\ -c_r(P_{tr}(i^*,d^*)) & \text{for } i = i^* \\ U_r'(x_r) - u_i & \text{for } i > i^* \end{cases}$$

We omit here the detailed proof. ■

In the next lemma, we derive sufficient conditions for determining whether a vector $(d_1^*,...,d_R^*)$ is an equilibrium allocation.

Lemma 4: Let Assumption 1, 2, and 3 hold. Then the allocation $(d_1^*,...,d_R^*)$ is an equilibrium allocation when the following properties hold for all users $r=1,...,R$.

(1) $d_r(i)=0$, for $i \notin \{i^*, i^*+1\}$.

(2) There exists a $\kappa = \in [0,1]$ such that

$$d_r(i^*) = \begin{cases} \frac{1}{P_{tr}(i^*,d)} D_r\left(\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)}\right), \\ \qquad \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} < u_{i^*+1}, \\ \frac{\kappa}{P_{tr}(i^*,d)} D_r\left(u_{i^*+1}\right), \\ \qquad \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} = u_{i^*+1}, \\ 0, \text{ otherwise,} \end{cases}$$

and

$$d_r(i^*+1) = \begin{cases} D_r\left(u_{i^*+1}\right), \\ \qquad \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} > u_{i^*+1}, \\ (1-\kappa) D_r\left(u_{i^*+1}\right) \\ \qquad \text{if } \frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} = u_{i^*+1}, \\ 0, \text{ otherwise.} \end{cases}$$

Proof: Note that when the Properties (1) and (2) hold, then Lemma 3 applies for each allocation d_r^*, $r=1,...,R$, and $(d_1^*,...,d_R^*)$ is an equilibrium allocation. In particular, Property (1) of Lemma 3 holds as we have that $d_r(i)=0$ for $i \notin \{i^*, i^*+1\}$. Furthermore, when $d_r^*(i^*) > 0$ and $\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} < u_{i^*+1}$, then we have that $d_r(i^*+1)=0$ and

$$x_r^* = P_{tr}(i^*) d_r(i^*) = D_r\left(\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)}\right).$$

It follows that

$$\begin{aligned} &U_r'(x_r^*)P_{tr}(i^*,d^*) - u_{i^*} - c_r(P_{tr}(i^*,d)) \\ &= \left(\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)}\right) P_{tr}(i^*,d^*) - u_{i^*} \\ &\quad -c_r(P_{tr}(i^*,d)) = 0, \end{aligned}$$

and Property (2) of Lemma 3 holds. Similarly, one can show that Property (2) of Lemma 3 holds when $d_r^*(i^*) > 0$ and $\frac{u_{i^*}+c_r(P_{tr}(i^*,d))}{P_{tr}(i^*,d)} = u_{i^*+1}$ and Property (3) of Lemma 3 is satisfied when $d_r^*(i^*+1) > 0$. Finally, Property (4) of Lemma 3 holds by the definition of the allocation vector d_r^*, $r=1,...,R$. ■

We can use Lemma 3 to prove Proposition 3.

Proof of Proposition 3: Property (a)-(d) of Proposition 3 can be proven by the same argument as the one used to prove Proposition 1. In particular, Property (2) of Lemma 3 implies that

$$P_{tr}(i^*,d^*) \geq \min_{r \in \{1,...,R\}} \frac{u_{i^*} - c_r(P_{tr}(i^*,d))}{u_{i^*+1}} \geq \frac{u_{i^*}}{u_{i^*+1}}.$$

We omit here the detailed proof. ■

We can use Lemma 4 to construct an equilibrium allocation and prove Proposition 4.

Proof of Proposition 4: Let the priority i_0 be such that

$$D(u_{i_0}) > C, \text{ and } D(u_{i_0+1}) \leq C.$$

Such a priority i_0 exists as by Assumption 1 and 2, we have that

$$D(u_1) > C \text{ and } D(u_{N+1}) = 0 < C.$$

For $\rho \in (0,1]$, let the allocations $d_r(\rho)$, $r=1,...,R$, be given by

$$d_r(i_0,\rho) = \begin{cases} \frac{1}{\rho} D_r\left(\frac{u_{i_0}+c_r(\rho)}{\rho}\right), & \text{if } \frac{u_{i_0}+c_r(\rho)}{\rho} \leq u_{i_0+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_r(i_0+1,\rho) = \begin{cases} D_r\left(u_{i_0+1}\right), & \text{if } \frac{u_{i_0}+c_r(\rho)}{\rho} > u_{i_0+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and $d_r(i,\rho)=0$ for $i \notin \{i_0, i_0+1\}$. Observe that $d_r(\rho)$ is right-continuous on $(0,1]$. Furthermore, $d_r(\rho)$ is continuous on $(0,1]$, except at the unique point ρ_r for which

$$\frac{u_{i_0}+c_r(\rho_r)}{\rho_r} = u_{i_0+1}.$$

Such a ρ_r exists as the function $\frac{u_{i_0}+c_r(\rho)}{\rho}$ is strictly decreasing and continuous on $(0,1]$, with

$$\lim_{\rho \uparrow 1} \frac{u_{i_0}+c_r(\rho)}{\rho} = u_{i_0} < u_{i_0+1}$$

and

$$\begin{aligned} \lim_{\rho \downarrow (u_{i_0}/u_{i_0+1})} \frac{u_{i_0}+c_r(\rho)}{\rho} &\geq \lim_{\rho \downarrow (u_{i_0}/u_{i_0+1})} \frac{u_{i_0}}{\rho} \\ &= u_{i_0+1}. \end{aligned}$$

Using this result, the function

$$P(\rho) = \frac{C - \sum_{r=1}^R d_r(i_0+1,\rho)}{\sum_{r=1}^R d_r(i_0,\rho)}$$

is continuous on $[u_{i_0}/u_{i_0+1}, 1)$ except at points ρ in the set $\mathcal{A} = \{\rho_r : r=1,...,R\}$. Let L be the cardinality of $\mathcal{A}$. Note that when there exist two users $r, r' \in \{1,...,R\}$, $r \neq r'$, for which we have that $\rho_r = \rho_{r'}$, then L is strictly less than R.

We define the vector α which lists the elements of $\mathcal{A}$ in ascending order, *i.e.* $\alpha = (\alpha(1), ..., \alpha(L)) \in \mathcal{A}^L$ is such that $\alpha(l) < \alpha(l')$ for $l < l'$. Using the vector α, we define the intervals $\mathcal{I}(l)$ by

$$\mathcal{I}(l) = [\alpha(l), \alpha(l+1)), \qquad l = 1, ..., L,$$

where we use the convention that $\alpha(L+1) = 1$. Note that $P(\rho)$ is continuous on $\mathcal{I}(l)$ for every $l = 1, ..., L$. Furthermore, observe that

$$\begin{aligned} \lim_{\rho \downarrow \alpha(1)} P(\rho) &= \frac{C - D(u_{i_0+1}) + \sum_{r:\rho_r = \alpha(1)} D_r(u_{i_0+1})}{(1/\alpha(1)) \sum_{r:\rho_r=\alpha(1)} D_r(u_{i_0+1})} \\ &= \alpha(1) \left(\frac{C - D(u_{i_0+1})}{\sum_{r:\rho_r=\alpha(1)} D_r(u_{i_0+1})} + 1 \right) \\ &\geq \alpha(1). \end{aligned}$$

It follows that there exists at least one $l \in \{1, ..., L\}$ such that $P(\alpha(l)) \geq \alpha(l)$. Let l^* be the largest such l. When

$$\lim_{\rho \uparrow \alpha(l^*+1)} P(\rho) < \alpha(l^*+1),$$

then we have that there exists a $\rho^* \in \mathcal{I}(l^*)$ such that $P(\rho^*) = \rho^*$. Note that in this case Lemma 4 applies to the allocation vector $d(\rho^*) = (d_r(\rho^*), ..., d_r(\rho^*))$; and $d(\rho^*) = (d_r(\rho^*), ..., d_r(\rho^*))$ is an equilibrium allocation.

Therefore, to prove Proposition 4, it suffices to consider the case where

$$\lim_{\rho \uparrow \alpha(l^*+1)} P(\rho) \geq \alpha(l^*+1).$$

Note that

$$P\Big(\alpha(L+1)\Big) = P(1) = \frac{C}{\sum_r D_r(u_{i_0})} < 1,$$

and it follows that $P(\alpha(l)) < \alpha(l)$ for $l = l^*+1, ..., L+1$. In particular, we have $P(\alpha(l^*+1)) < \alpha(l^*+1)$. For $\rho^* = \alpha(l^*+1)$ and $\kappa \in [0,1]$, let the allocations $d_r(\rho^*, \kappa)$, $r = 1, ..., R$, be given by

$$d_r(i_0, \rho^*, \kappa) = \begin{cases} \frac{\kappa}{\rho^*} D_r\left(u_{i_0+1}\right), & \text{if } \rho_r = \rho^*, \\ d_r(i_0, \rho^*) & \text{otherwise,} \end{cases}$$

$$d_r(i_0+1, \rho^*, \kappa) = \begin{cases} (1-\kappa) D_r\left(u_{i_0+1}\right), & \text{if } \rho_r = \rho^*, \\ d_r(i_0+1, \rho^*) & \text{otherwise,} \end{cases}$$

and $d_r(i, \rho^*, \kappa) = 0$ for $i \notin \{i_0, i_0+1\}$. Note that the allocations $d_r(\rho^*, \kappa)$, $r = 1, ..., R$, and the function

$$P(\rho^*, \kappa) = \frac{C - \sum d_r(i_0+1, \rho^*, \kappa)}{\sum d_r(i_0, \rho^*, \kappa)},$$

are continuous in κ on $[0,1]$. Furthermore, we have that

$$\lim_{\kappa \uparrow 1} P(\rho^*, \kappa) = \lim_{\rho \uparrow \alpha(l^*+1)} P(\rho)$$

and

$$\lim_{\kappa \downarrow 0} P(\rho^*, \kappa) = P(\alpha(l^*+1)).$$

As we have that

$$\lim_{\rho \uparrow \alpha(l^*+1)} P(\rho) \geq \alpha(l^*+1) \quad \text{and} \quad P(\alpha(l^*+1)) < \alpha(l^*+1),$$

it follows that there exists a $\kappa^* \in [0,1]$ such that

$$P(\rho^*, \kappa^*) = \alpha(l^*+1).$$

Note that Lemma 4 applies to the allocation vector $d(\rho^*, \kappa^*) = (d_r(\rho^*, \kappa^*), ..., d_r(\rho^*, \kappa^*))$, and $d(\rho^*, \kappa^*) = (d_r(\rho^*, \kappa^*), ..., d_r(\rho^*, \kappa^*))$ is therefore an equilibrium allocation. This completes the proof.

■