Trees as Semilattices

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Abstract

We study semilattices whose diagrams are trees. First, we characterize them as semilattices whose convex subsemilattices form a convex geometry, or, equivalently, the closure induced by convex subsemilattices is antiexchange. Then we give lattice theoretic and two graph theoretic characterizations of atomistic semilattices with tree diagrams.

1 Introduction

Graph theoretic properties of lattice and semilattice diagrams are of great interest in lattice theory in combinatorics. Even such fundamental properties of lattices as distributivity and modularity can be expressed as properties of diagrams. Various graph theoretic properties of diagrams give rise to very interesting classes of lattices. For example, planar lattices were characterized in [7] via a number of forbidden configurations. A simple forbidden configuration, a poset with the diagram like the letter N, has a nice characterization for posets which generalizes smoothly to lattices and semilattices [4, 12, 9]. In this paper we look at a very simple property of a poset diagram — we study finite posets whose diagrams are rooted trees. Such posets are semilattices because unique paths from any two nodes to the root have a minimal common point which is the least upper bound. Chains being the only exception, lattice diagrams are not trees, but a similar investigation for lattices can be carried out if only non-zero elements are considered. However, lattices whose non-zero elements have a tree diagram are equivalent to tree diagram semilattices.

The paper is organized in three sections. In the remainder of this section we give all necessary definitions. In Section 2 we characterize tree-diagram semilattices as semilattices having antiexchange closures induced by their convex subsemilattices. Families of closed sets of antiexchange closures are known under the name of *convex geometries* and families of complements of closed sets are sometimes

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referred to as *antimatroids*, see [1, 2, 3, 8]. It is well-known that the closure operator induced by subsemilattices of a semilattice is antiexchange. If the family of subsemilattices is restricted to the convex ones, then the antiexchange property gives us tree-diagram semilattices.

In Section 3 atomistic tree diagram semilattices are studied. Three characterizations are obtained. Firstly, it is shown that such semilattices are exactly series-parallel atomistic semilattices. Secondly, trees arising as diagrams of such semilattices are characterized as branchy trees, i.e. trees whose vertices, except for leaves, have at least two children. Finally, it is observed that tree-diagram semilattices can be described by complete chromatic graphs with four forbidden subgraphs.

In the sequel, lattices and semilattices will be denoted by the letters L and S respectively (possibly with indices) and 0 and 1 will stand for the least and the greatest elements. In this paper we consider only finite lattices and semilattices. The semilattices are join-semilattices, that is, the order is given by $x \leq y \Leftrightarrow x \vee y = y$. Graphs will be denoted by $\langle V, E \rangle$, where V is a set of vertices and E a set of edges. A tree with a root s will be denoted by $\langle V, E, s \rangle$.

A semilattice is called *tree-diagram* if its diagram is a rooted tree with root 1. In the sequel we shall always assume that whenever the diagram of a semilattice is a tree, it is rooted and the root is the maximal element. This corresponds to the definition of a *computer science tree* in [13]. In [13], a *poset tree* is a poset whose cover graph is a tree (that is, does not contain a circuit). Generally, a poset tree may not be a computer science tree; however, in the case of finite semilattices, these two definitions are equivalent.

Below all other definitions are given.

Tree-diagram lattice : A lattice L such that the diagram of the join-semilattice $L - \{0\}$ is a tree.

Series-parallel poset : A poset containing no four-element subposet with diagram like the letter N.

Series-parallel lattice (semilattice) [9] : A lattice (semilattice) which is series-parallel as a poset.

Antiexchange closure [2] : A closure G on a set X satisfying:

$$\forall A \subseteq X, x, y \in X : x, y \notin G(A), x \in G(A \cup y) \Rightarrow y \notin G(A \cup x).$$

Convex geometry [3] : A family of closed sets of an antiexchange closure.

$Sub(S)$ (or $Sub(L)$) : The family of all subsemilattices (or sublattices) of S (or L).

$CSub(S)$: The family of all convex, or order preserving subsemilattices of S, that is, subsemilattices S' such that $x \leq y \leq z$ and $x, z \in S'$ imply $y \in S'$.

Ordinal sum of posets $\langle P_1, \leq_1 \rangle$ *and* $\langle P_2, \leq_2 \rangle$ *with* $P_1 \cap P_2 = \emptyset$: The poset $\langle P_1 \cup P_2, \leq \rangle$ where $\leq$ coincides with $\leq_1$ and $\leq_2$ on P_1 and P_2, and if $p_1 \in P_1, p_2 \in P_2$ then $p_1 \leq p_2$. This poset is denoted by $P_1 \oplus P_2$.

Single-element poset will be denoted by $\mathbf{1}$ and $\mathbf{2}$ stands for a two-element chain.

Branchy tree : A rooted tree $\langle V, E, s \rangle$ such that $val(s) \neq 1$ and for all $v \in V - s : val(v) \neq 2$, where $val(v) = \mid \{w \in V : (v, w) \in E\} \mid$ (i.e. all vertices that are not leaves have at least two children).

Atomistic lattice : A lattice every non-zero element of which is the join of atoms.

Atomistic semilattice: A semilattice every element of which is the join of the minimal elements below it. If L is atomistic, then so is $L - \{0\}$ considered as a join-semilattice.

2 Tree-diagram lattices and semilattices and the antiexchange closures

In this section we first show that tree-diagram lattices are of form $L \simeq \mathbf{1} \oplus S$, where S is a tree-diagram semilattice. Therefore, all results about tree-diagram semilattices can be reformulated for tree-diagram lattices in a straightforward manner. Then we prove the main result of the section stating that a semilattice S is tree-diagram iff $CSub(S)$ is a convex geometry.

We start with a simple lemma whose proof is omitted.

Lemma 1 *Let S be a tree-diagram semilattice and S' its subsemilattice with the least element x. Then S' is a chain. In particular, a lattice L is tree-diagram iff $L \simeq \mathbf{1} \oplus S$ for a tree-diagram semilattice S.* $\square$

Therefore, it suffices to prove all results for tree-diagram semilattices only. Now we are ready to prove the main result of this section.

Theorem 1 *A semilattice S is tree-diagram iff $CSub(S)$ is a convex geometry.*

Proof. Let S be a tree-diagram semilattice. To prove that $CSub(S)$ is a convex geometry, we must show that for every $S' \in CSub(S)$ if $S' \neq S$ then there exists $x \notin S'$ such that $S' \cup x \in CSub(S)$ (see equivalent definitions of convex geometry in [3]). If S has unique minimal element, it is a chain by lemma 1 and its intervals form a convex geometry [3]. Suppose S has two or more minimal elements. Two cases arise.

Case 1. S' contains all minimal elements of S. If $S' = S$, then we are done. If $S' \neq S$, consider the top element y of S', which does not coincide with 1 because S' is convex. Then its filter, $[y)$, has at least two elements, and since $[y)$ is a subsemilattice, by lemma 1, y is covered by a unique element x. Prove that $S' \cup x \in CSub(S)$. Clearly, $S' \cup x \in Sub(S)$. It is enough to prove that $b \in S'$ whenever $a < b < x$ for $a \in S'$. Let c be a minimal element of S such that $b > c$. Then $[c)$ is a chain and $y > c$. Hence either $b \geq y$ or $y \geq b$. By the definitions of x and b, $y \geq b$. Therefore, $b \in S'$ because $S' \in CSub(S)$. Thus, $S' \cup x \in CSub(S)$.

Case 2. There is a minimal element of S which does not belong to S'. Then there is an element $x \notin S'$ covered by $y \in S'$. Prove that $S' \cup x \in CSub(S)$. Let $z \in S'$. Then $z \vee y \geq z \vee x$. Since $[x)$ is a chain and y covers x, we have that $z \vee x \geq y$ and $z \vee x = z \vee y \in S'$. Hence, $S' \in Sub(S)$. Let $x < z < v \in S'$. Since $[x)$ is a chain, y is the unique cover of x and $z \geq y$. Since S' is order preserving, so is $S' \cup x$. Therefore, $CSub(S)$ is a convex geometry.

Conversely, assume that S is a semilattice whose diagram S is not a tree. Consider a circuit on this diagram. Let x be a minimal element of this circuit and y, z its neighbors. Then both y and z cover x. Let $p = y \vee z$. Then $p \neq y, z$ and the minimal order preserving subsemilattice containing $\{x, p\}$ or $\{x, y, z\}$ is $[x, p]$. Then, according to the list of the equivalent definitions of convex geometries [3], $CSub(S)$ is not convex geometry because in a convex geometry no set may have two different bases. The theorem is completely proved. $\square$

There is another relationship between tree diagrams and convex geometries: if a lattice L is tree-diagram, then $Sub(L)$ is a convex geometry. Indeed, a tree-diagram lattice is series-parallel (having an N would imply having a circuit) and $Sub(L)$ is a convex geometry iff L is series-parallel [9, 11].

We have seen that in a tree-diagram semilattice two incomparable elements can not have a common lower bound. Therefore, if $x = a_1 \vee \ldots \vee a_n$ in a tree-diagram semilattice, $x = a_i \vee a_j$ for appropriate $i, j \in \{1, \ldots, n\}$. Tree diagram lattices or semilattices are planar and hence have dimension one or two. Either of these facts implies that tree-diagram lattices are 2-distributive, that is, they satisfy $x \wedge (y_0 \vee y_1 \vee y_2) = (x \wedge (y_0 \vee y_1)) \vee (x \wedge (y_0 \vee y_2)) \vee (x \wedge (y_1 \vee y_2))$, cf. [10].

The structure of modular and distributive tree-diagram lattices can be easily described. Let M_n be an $n-$point projective line, i.e. $M_n = \{0, 1, a_1, \ldots, a_n\}$ where $a_i \vee a_j = 1, a_i \wedge a_j = 0$ whenever $i \neq j$.

Proposition 1 *A lattice L is modular and tree-diagram iff $L \simeq L_1 \oplus L_2$ where L_1 is either isomorphic to M_n for some n or empty, and L_2 is a chain.*

Proof. The *'if'* part is obvious. To prove the *'only if'* part, let L be modular tree-diagram lattice. If $|L| = 1$, we are done. Let $|L| > 1$. Let $a_1, \ldots, a_n$ be atoms of L, $n \geq 1$. Suppose $a = a_1 \vee \ldots \vee a_n$. If there is $b \| a$, then, by lemma 1, $b \| a_1$ and $b \wedge a = 0$. $b \wedge a_1 = 0$ because a_1 is an atom. Since $b \vee a_1$ and a can not be incomparable in view of lemma 1, $b \vee a_1 \geq a$. This means that $b \vee a_1 = b \vee a$. Therefore, $\{0, a_1, a, b, a \vee b\}$ is a sublattice of L isomorphic to N_5. This contradiction shows that $L \simeq [0, a] \oplus L_2$ where L_2 is a chain by lemma 1.

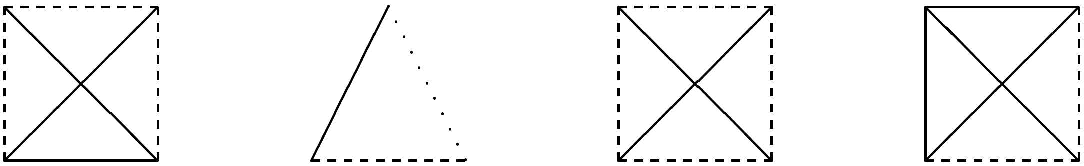
If $n = 1$ then L is a chain, and L_1 is empty. If $n \neq 1$, we have to prove that $[0, a] \simeq M_n$. To do this, we only have to show that a covers a_i for all $i \in [1, n]$. Suppose there is such i that a does not cover a_i, i.e. $a > b > a_i$ for some b. Since $b < a$, there is $a_j \not\leq b$. Let $x = a_j \vee b$. Then $x \geq a_i$ and by lemma 1 x and $y = a_i \vee a_j$ are comparable. If y were less than b, we would have $a_j \leq b$. Hence, $y \geq b$ and therefore $x = y$. It shows that $\{0, a_i, a_j, b, x\}$ is a sublattice isomorphic to N_5. This contradiction proves that a covers a_i, i.e. $[0, a] \simeq M_n$ and $L \simeq M_n \oplus L_2$ where L_2 is a chain. Proposition is proved. $\Box$

Corollary 1 *A lattice L is distributive and tree-diagram iff $L \simeq L_1 \oplus L_2$ where L_1 is either empty or isomorphic to* **2** *or* **2**×**2** *and L_2 is a chain.* $\Box$

3 Atomistic tree-diagram semilattices and chromatic graphs

In this section we characterize atomistic tree-diagram semilattices as atomistic series-parallel semilattices and show that their diagrams are branchy trees. Then we extend the representation technique for positional structures in game theory (cf. [5, 6]) to describe such semilattices via complete chromatic subgraphs with four forbidden subgraphs.

Let $K = \langle A, E \rangle$ be a finite complete graph without loops and multiple edges, i.e. $E = \{(a_1, a_2) : a_1, a_2 \in A, a_1 \neq a_2\}$. Let $c : E \to N$ be a coloring mapping. Usually $N = \{1, 2, \ldots, n\}$, i.e. edges are colored with n colors : $c(a_1, a_2) \in N$ is the color of the edge $(a_1, a_2) \in E$. Such a triple $\Gamma = \langle N, A, c \rangle$ is called a *chromatic graph*. Each subset $A' \subseteq A$ generates a *chromatic subgraph* Γ' of Γ. In what follows, four chromatic subgraphs $\Pi, \Delta, \Box, \Uparrow$ depicted on the figure below will pay the crucial role.



Π Δ $\square$ ⍏

Given a semilattice S, let A_S be the set of its atoms. Define a chromatic graph Γ_S associated with S as follows: $\Gamma_S = \langle S, A_S, c_S \rangle$ where $c_S(x, y) = x \vee y \in S$.

Theorem 2 *Given a semilattice S, the following are equivalent:*
1) *S is atomistic and series-parallel;*
2) *S is atomistic and tree-diagram;*
3) *The diagram of S is a branchy tree with root 1.*
In addition, if S is atomistic, then it is tree-diagram iff the chromatic graph Γ_S does not contain subgraphs isomorphic to $\Pi, \Delta, \square,$⍏.

Proof. **1)** $\Rightarrow$ **3)**. Let S be atomistic and series-parallel. Prove that S is tree-diagram first. Assume it is not and consider a circuit with a minimal element x and its neighbors y, z. Both y and z cover x. Since S is atomistic, there is an atom $a \leq y$ such that $a \not\leq z$, and hence $a \not\leq x$. Therefore, $a < y, y > x, x < z$ ($a \neq y$ because y is not an atom) and $a \| x, a \| z, y \| z$. Thus, S is not series-parallel. This contradiction shows that S is tree-diagram. Show that the diagram of S considered as a rooted tree with root 1 is branchy. Suppose there is an element $x \neq 1$ with $val(x) = 2$, that is, x covers a unique element y, because x is covered by a unique element by lemma 1. Consider an atom $a \leq x$ such that $a \not\leq y$. Clearly, $x \neq a$ for x is not an atom because atoms are terminal vertices of the considered rooted tree, and for every atom $b : val(b) = 1$. Hence, there exists $z \in [a, x]$ covered by x, and since x covers only y, $z = y$. Thus, $y \geq a$, which contradicts our assumption. Hence, $val(x) \neq 2$ for all $x \neq 1$. If $val(1) = 1$, then let x be the only element covered by 1 and let a be an atom. There exists an element y covered by 1 in $[a, 1]$ and, since x is the only element covered by 1, $x = y \geq a$. Therefore, x is greater than the join of all atoms and 1 can not be represented as the join of atoms. This contradiction shows $val(1) \neq 1$ and finishes the proof of **1)** $\Rightarrow$ **3)**.

3) $\Rightarrow$ **2)**. Let **3)** hold. Then S is tree-diagram and we must prove that S is atomistic. Let x be a join-irreducible element which is not an atom. Then x covers a unique element. If $x = 1$, then $val(1) = 1$, and if $x \neq 1$, then, by lemma 1, x has a unique cover and $val(x) = 2$, i.e. the diagram of S is not branchy. This contradiction shows that S is atomistic.

That **2)** implies **1)** follows from the fact that any tree-diagram semilattice is series-parallel.

To prove the last statement, we need a few auxiliary definitions. Let $\mathcal{G} = \langle T, N, \varphi \rangle$, where T is a rooted tree $\langle V, E, s \rangle$ whose set of leaves is denoted by A, N is a finite set and φ is a map from $V - A$ to N. (These constructions are called positional structures in game theory). Associate a chromatic graph $\Gamma = \tau(\mathcal{G}) = \langle N, A, c \rangle$ with $\mathcal{G}$, where the coloring function is defined as follows. If (a_i, a_j) is an edge in Γ, let p_{ij} be the common node of paths s–a_i and s–a_j which is farthest from the root. Then $c(a_i, a_j) = \varphi(p_{ij})$. For example, if T is a two-colored balanced binary tree of depth 2, whose root is colored by one color and intermediate nodes by the other, then τ applied to it would yield a chromatic graph isomorphic to $\square$. We call $\mathcal{G}$ nonrepeated if $\varphi(b) \neq \varphi(b')$ whenever (b, b') is an edge in T. It was proved by the second author in [5, 6] that the mapping τ is a 1-1 correspondence between nonrepeated structures $\mathcal{G}$ whose underlying trees are branchy, and chromatic graphs without subgraphs isomorphic to Π and Δ.

Moreover, if φ is injective, then $\tau(\mathcal{G})$ does not contain a subgraph isomorphic to $\square$ or $\Uparrow$ [5]. Conversely, if Γ is chromatic graph not containing subgraphs isomorphic to $\Pi, \Delta, \square$ and $\Uparrow$, by the result cited above there exists a unique nonrepeated structure $\mathcal{G}$ whose underlying tree is branchy such that $\tau(\mathcal{G}) = \Gamma$. Prove that φ of that structure $\mathcal{G}$ is injective. Suppose it is not, that is, $\varphi(b) = \varphi(b')$ for $b \neq b'$. Since $\mathcal{G}$ is nonrepeated, b and b' are not adjacent. Then there exists a node a inside the path b–b' such that $\varphi(a) \neq \varphi(b)$. Two cases arise depending on whether there is a path from the root containing both b, b'. It is easy to show that in the first case when such path exists, $\tau(\mathcal{G})$ contains a chromatic subgraph isomorphic to $\Uparrow$, and in the second case when there is no such path, $\tau(\mathcal{G})$ contains a chromatic subgraph isomorphic to either $\Uparrow$ or $\square$. Therefore, we have proved that the mapping τ establishes a 1-1 correspondence between nonrepeated structures $\mathcal{G}$ with injective functions φ and whose underlying trees are branchy and chromatic graphs without chromatic subgraphs isomorphic to $\Pi, \Delta, \square, \Uparrow$.

Now, given a tree-diagram semilattice S, consider $\mathcal{G}_S = \psi(S) = \langle T_S, S - A_S, id\rangle$, where T_S is the diagram of S. The semilattice S is tree-diagram iff T_S is branchy. Therefore, since $\Gamma_S = \tau(\psi(S))$, the one-to-one correspondence established above finishes the proof of the theorem. $\square$

Corollary 2 *For any tree-diagram semilattice S, the chromatic graph Γ_S does not contain subgraphs isomorphic to $\Pi, \Delta, \square, \Uparrow$. Moreover, the mapping $S \longrightarrow \Gamma_S$ is a 1-1 correspondence between atomistic tree-diagram semilattices and chromatic graphs without subgraphs isomorphic to $\Pi, \Delta, \square, \Uparrow$.* $\square$

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