Game-based Notions of Locality over Finite Models

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Abstract. Locality notions in logic say that the truth value of a formula can be determined locally, by looking at the isomorphism type of a small neighborhood of its free variables. Such notions have proved to be useful in many applications. They all, however, refer to isomorphism of neighborhoods, which most local logics cannot test for. A more relaxed notion of locality says that the truth value of a formula is determined by what the logic itself can say about that small neighborhood. Or, since most logics are characterized by games, the truth value of a formula is determined by the type, with respect to a game, of that small neighborhood. Such game-based notions of locality can often be applied when traditional isomorphism-based locality cannot.

Our goal is to study game-based notions of locality. We work with an abstract view of games that subsumes games for many logics. We look at three, progressively more complicated locality notions. The easiest requires only very mild conditions on the game and works for most logics of interest. The other notions, based on Hanf's and Gaifman's theorems, require more restrictions. We state those restrictions and give examples of logics that satisfy and fail the respective game-based notions of locality.

1 Introduction

Locality is a property of logics that finds its origins in the work by Hanf [13] and Gaifman [10], and that was shown to be very useful in the context of finite model theory. Locality is primarily used in two ways: for proving inexpressibility results, and for establishing normal forms for logical formulae. The former has led to new easy winning strategies in logical games [6, 8, 20], with applications in descriptive complexity (e.g., the study of monadic NP and its relatives [8], or circuit complexity classes [21]), in databases (e.g., establishing bounds on the expressiveness of aggregate queries [16], or on query rewriting in data integration and exchange [7, 1]), and in formal languages (e.g., in characterizing subclasses of star-free languages [27]). Local normal forms like those in [10, 25] have found many applications as well, for example, in the design of low-complexity model-checking algorithms [9, 12, 26], in automata theory [25] and in computing weakest preconditions for database transactions [2].

There are two closely related ways of stating locality of logical formulae. One, originating in Hanf's work [13], says that if two structures $\mathfrak{A}$ and $\mathfrak{B}$ realize the same multiset of isomorphism types of neighborhoods of radius d, then they agree on a given sentence Φ. Here d depends only on Φ.

The notion of locality inspired by Gaifman's theorem [10] says that if the d-neighborhoods of two tuples $\bar{a}_1$ and $\bar{a}_2$ in a structure $\mathfrak{A}$ are isomorphic, then $\mathfrak{A} \models \varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$. Again, d depends on φ, and not on $\mathfrak{A}$.

If all formulae in a logic are local, it is easy to prove bounds on its expressive power. For example, connectivity violates the Hanf notion of locality, as one cycle of length $2m$ and two disjoint cycles of length m realize the same multiset of isomorphism types of neighborhoods of radius d as long as $m > 2d + 1$. Likewise, the transitive closure of a graph violates the Gaifman notion of locality: in the graph in Fig. 1, one can find two elements a, b such that the radius-d neighborhoods of (a, b) and (b, a) are isomorphic, and yet the transitive closure distinguishes these tuples.

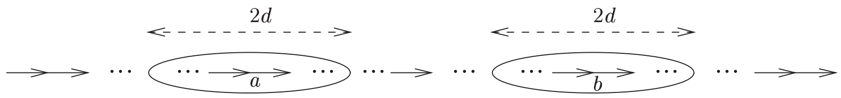

Fig. 1. Locality and transitive closure.

These notions of locality, while very useful in many applications, have one deficiency: they all refer to *isomorphism* of neighborhoods, which is a very strong property (typically not expressible in a logic that satisfies one of the locality properties). There are situations when these notions are not applicable simply because structures do not have enough isomorphic neighborhoods! One example was given in [21] which discussed applicability of locality techniques to the study of small parallel complexity classes: consider a directed tree in which all non-leaf nodes have different out-degrees. Then locality techniques cannot be used to derive any results about logics over such trees.

Intuitively, it seems that requiring isomorphism of neighborhoods is too much. Suppose we are dealing with first-order logic FO, which is local in the sense of Gaifman. For a structure $\mathfrak{A}$, it appears that if FO itself cannot see the difference between two large enough neighborhoods of points a and b in $\mathfrak{A}$, then it should not be able to see the difference between elements a and b in $\mathfrak{A}$. That is, for a given formula $\varphi(x)$, if radius-d neighborhoods of a and b cannot be distinguished by sufficiently many FO formulae, then $\mathfrak{A} \models \varphi(a) \leftrightarrow \varphi(b)$.

Gaifman's theorem [10] actually implies that this is the case: if φ is of quantifier rank k, then there exist numbers d and l, dependent on k only, such that if radius-d neighborhoods of a and b cannot be distinguished by formulae of quantifier rank l, then $\mathfrak{A} \models \varphi(a) \leftrightarrow \varphi(b)$.

In fact, it seems that if a logic is local (say, in the sense of Gaifman), then for each formula φ there is a number d such that if the logic cannot distinguish radius-d neighborhoods of $\bar{a}$ and $\bar{b}$, then $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$.

The goal of this paper is to introduce such notions of locality based on logical indistinguishability of neighborhoods, and see if they apply to logics that are known to possess isomorphism-based locality properties. Since logical equivalence is often captured by Ehrenfeucht-Fraïssé-type of games, we shall refer to such new notions of locality as *game-based.*

We shall discover that the situation is more complex than one may have expected, and passing from isomorphism-based locality to game-based is by no means guaranteed for logics known to possess the former.

To be able to talk about general game-based locality notions, we need a unifying framework for talking about logical games that subsumes games for FO, and many of its counting and generalized quantifier extensions. A game is described via an *agreement*, which is a collection of *tactics*, and each tactic is a set of partial functions according to which the game is played. We present this framework in Section 3.

To analyze game-based locality, we then study conditions on agreements that guarantee one of the locality notions. We look at three progressively more complex notions: weak locality, Gaifman-locality, and Hanf-locality, which are described in Section 4. Weak locality is a variation of Gaifman-locality that applies to non-overlapping neighborhoods. In general, establishing some variation of game-based locality for a logic $\mathcal{L}$ does *not* imply that fragments or extensions of $\mathcal{L}$ will possess the same locality property.

Weak locality turns out to require very little and holds for so-called *basic* agreements, as shown in Section 6. While most games of interest are based on basic agreements, we give an example of one unary generalized-quantifier extension of FO which fails weak locality.

In Section 7, we study Hanf-locality under games and show that it holds for a class of agreements that we call matching. These include games for some counting extensions of FO, but game-based Half-locality fails for FO itself (as was already observed in [25]) and some of its generalized-quantifier extensions.

In Section 8, we study Gaifman-locality under games. We show that this notion often implies a normal form result for a logic, similar to Gaifman's theorem for FO. We establish Gaifman-locality for games corresponding to FO and some of its extensions.

Due to space limitations, proofs are not presented in this extended abstract. A full version containing all the proofs can be obtained from the authors.

2 Notation

We work with finite structures, whose universes are subsets of some countable infinite set U. All vocabularies will be finite sequences of relation symbols $\sigma = \langle R_1, \ldots, R_n \rangle$; a σ-structure $\mathfrak{A}$ consists of a finite universe $A \subset U$ and an interpretation of each m-ary relation symbol R_i in σ as a subset of A^m. We adopt the convention that the universe of a structure is denoted by the corresponding Roman letter, that is, the universe of $\mathfrak{A}$ is A, the universe of $\mathfrak{B}$ is B, etc. Isomorphism of structures will be denoted by $\cong$.

For a relation $F \subseteq A \times B$, we use $\mathrm{dom}(F)$ to denote its domain $\{a \in A \mid \exists b \ (a, b) \in R\}$ and $\mathrm{rng}(F)$ to denote the range $\{b \in B \mid \exists a \ (a, b) \in R\}$. We use the same notation dom and rng for the domain and range of a (partial) function. The graph of a function $f : A \to B$ is denoted by $\mathrm{graph}(f) = \{(a, b) \mid b = f(a)\}$.

Given two tuples $\bar{a}_1$ and $\bar{a}_2$, we write $\bar{a}_1\bar{a}_2$ for their concatenation.

Next, we introduce the logics considered in the paper. First-order logic will be denoted by FO. Then, we define *simple unary generalized quantifiers* $\mathbf{Q}_S$ [19, 28]. Let $S \subseteq \mathbb{N}$. We denote by $\mathrm{FO}(\mathbf{Q}_S)$ the extension of FO with the following formation rule: if $\psi(x, \bar{y})$ is a formula, then $\varphi(\bar{y}) = \mathbf{Q}_S x\ \psi(x, \bar{y})$ is a formula. The semantics is as follows: $\mathfrak{A} \models \varphi(\bar{a})$ if $|\{a \mid \mathfrak{A} \models \psi(a, \bar{a})| \in S$. One could also define FO extended with a collection of simple unary generalized quantifiers.

We consider one special case of unary quantifiers: *modulo quantifiers* (cf. [23, 24, 28]). If $S = \{n \cdot p \mid n \in \mathbb{N}\}$, then we write $\mathbf{Q}_p$ instead of $\mathbf{Q}_S$.

Finally, we define a powerful counting logic that subsumes most counting extensions of FO, in particular FO extended with arbitrary collections of unary generalized quantifiers. The structures for this logic are two-sorted, being $\mathbb{N}$ the second sort. There is a constant symbol for each $k \in \mathbb{N}$. The logic has *infinitary* connectives $\bigvee$ and $\bigwedge$, and *counting terms*: if φ is a formula and $\bar{x}$ a tuple of free first-sort variables in φ, then $\#\bar{x}.\varphi$ is a term of the second sort, whose free variables are those in φ except $\bar{x}$. Its value is the number of tuples $\bar{a}$ that make $\varphi(\bar{a}, \cdot)$ true. This logic, denoted by $\mathcal{L}_{\infty\omega}(\mathbf{Cnt})$, defines all properties of finite structures.

To restrict it, we use the notion of quantifier rank $\mathsf{qr}(\cdot)$ which is defined as the maximum depth of quantifier nesting (*excluding* quantification over the numerical universe for two-sorted logics). For $\mathcal{L}_{\infty\omega}(\mathbf{Cnt})$, we also define $\mathsf{qr}(\#\bar{x}.\varphi)$ as $\mathsf{qr}(\varphi) + |\bar{x}|$.

We now define $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ as $\mathcal{L}_{\infty\omega}(\mathbf{Cnt})$ restricted to formulae and terms that have finite rank. This logic subsumes known counting extensions of FO, but cannot express many properties definable, say, in fixed-point logics or fragments of second-order logic [20].

3 Games and logics

We now present the first way of abstractly viewing games such as Ehrenfeucht-Fraïssé games, as well as games for counting and unary-quantifier extensions of FO. Such games are played by two players, the *spoiler* and the *duplicator*, on two σ-structures $\mathfrak{A}$ and $\mathfrak{B}$. The goal of the spoiler is to show that the structures are different while the duplicator is trying to show that they are the same.

In most games, the spoiler and the duplicator agree on a class of relations before the game starts, that is, for each $A, B \subset U$, they have sets $\mathfrak{F}(A, B) = \{\mathcal{F}_1(A, B), \ldots, \mathcal{F}_s(A, B)\}$, where each $\mathcal{F}_i(A, B)$ is a a family of subsets of $A \times B$. The game starts with a position $(\bar{a}_0, \bar{b}_0)$, where $\bar{a}_0 \in A^l, \bar{b}_0 \in B^l$ (l could be 0). After i rounds, the *position* of the game consists of $(\bar{a}_0, a_1, \ldots, a_i)$ in $\mathfrak{A}$ and $(\bar{b}_0, b_1, \ldots, b_i)$ in $\mathfrak{B}$. Given a position $(\bar{a}_0\bar{a},\ \bar{b}_0\bar{b})$ after round i, the game proceeds as follows:

1. The spoiler selects a structure, $\mathfrak{A}$ or $\mathfrak{B}$.

2. The duplicator picks a family of relations $\mathcal{F}(A,B) \in \mathfrak{F}(A,B)$, if the spoiler selected $\mathfrak{A}$, or $\mathcal{F}(B,A) \in \mathfrak{F}(B,A)$, if the spoiler selected $\mathfrak{B}$. Assume that the spoiler chose $\mathfrak{A}$, the other case being completely symmetric.
3. The spoiler chooses one relation $F \in \mathcal{F}(A,B)$ and an element $a \in \mathrm{dom}(F)$.
4. The duplicator responds with an element $b \in \mathrm{rng}(F)$ such that $(a,b) \in F$, and the game continues from the position $(\bar{a}_0\bar{a}a,\ \bar{b}_0\bar{b}b)$.

We now present games corresponding to FO, $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$, and $\mathrm{FO}(\mathbf{Q}_p)$.

- If $\mathfrak{F}(A,B) = \{\{A \times B\}\}$ for every $A,B \subset U$, then this is the usual Ehrenfeucht-Fraïssé game: the spoiler is free to choose any point in A, and the duplicator is free to choose any point in B.
- Let $f_1,\ldots,f_r$ enumerate all the bijections $A \to B$. Suppose $\mathfrak{F}(A,B) = \{\{\mathrm{graph}(f_1)\},\ldots,\{\mathrm{graph}(f_r)\}\}$. Then we have the *bijective* game of [14]. In this game, in each round, the duplicator selects a bijection $f : A \to B$; the spoiler plays $a \in A$ and the duplicator responds by $f(a) \in B$.
- Given $A,B \subset U$, consider sets $\mathcal{F}(A,B)$ of the form $\{C_i \times D_i \mid C_i \subseteq A, D_i \subseteq B, i \in I\}$, where every subset of B occurs as one of the D_i's, and $|C_i| = |D_i| (\mathrm{mod}\ p)$ for each i. $\mathfrak{F}(A,B)$ consists of all $\mathcal{F}(A,B)$'s of this form. This is the setting of the game for modulo p quantifiers $\mathbf{Q}_p$ [24]. In each round of this game, the spoiler chooses $D \subseteq B$ and the duplicator selects $C \subseteq A$ with $|C| = |D| (\mathrm{mod}\ p)$. Then the spoiler plays $a \in A$ and the duplicator responds with $b \in B$ such that $a \in C$ iff $b \in D$.

The presentation of games given above is standard in the literature. For stating results in the paper, we shall use a slightly different way of presenting games. Suppose we have a position $(\bar{a}_0\bar{a},\ \bar{b}_0\bar{b})$ in the game, and the duplicator chooses a family $\mathcal{F}(A,B) \in \mathfrak{F}(A,B)$. By doing so, the duplicator is certain that, no matter what relation $F \in \mathcal{F}(A,B)$ the spoiler chooses, for every $a \in \mathrm{dom}(F)$, he has a response $b \in \mathrm{rng}(F)$. That is, for every $F \in \mathcal{F}(A,B)$, the duplicator has one or more *functions* $f : A \to B$ with $\mathrm{graph}(f) \subseteq F$, such that if the spoiler plays $a \in A$, he can respond with $f(a) \in B$. From now on, we shall be defining games using such a functional approach.

Definition 1. *An* agreement *is a collection* $\mathfrak{F} = \{\mathfrak{F}(A,B) \mid A,B$ *are finite subsets of* $U\}$, *where each* $\mathfrak{F}(A,B)$ *is of the form* $\{\mathcal{F}_1(A,B),\ldots,\mathcal{F}_m(A,B)\}$, $m \geq 0$, *and each* $\mathcal{F}_i(A,B)$ *is a nonempty collection of partial functions* $f : A \to B$. *We shall call the sets* $\mathcal{F}_i(A,B)$*'s* tactics.

The $\mathfrak{F}$-game *on* $(\mathfrak{A},\bar{a}_0)$ *and* $(\mathfrak{B},\bar{b}_0)$ *is played as follows. Suppose after* i *rounds the position is* $(\bar{a}_0\bar{a},\bar{b}_0\bar{b})$ *(before the game starts, the tuples* $\bar{a},\bar{b}$ *are empty). Then, in round* $i+1$:

1. *The spoiler chooses a structure,* $\mathfrak{A}$ *or* $\mathfrak{B}$. *Below we present the moves assuming he chose* $\mathfrak{A}$, *the case of* $\mathfrak{B}$ *is symmetric.*
2. *The duplicator chooses a tactic* $\mathcal{F}(A,B) \in \mathfrak{F}(A,B)$.
3. *The spoiler chooses a partial function* $f \in \mathcal{F}(A,B)$ *and an element* $a \in \mathrm{dom}(f)$; *the game continues from the position* $(\bar{a}_0\bar{a}a,\ \bar{b}_0\bar{b}f(a))$.

The duplicator wins after k-rounds if $\mathfrak{F}(A, B) \neq \emptyset$ and $\mathfrak{F}(B, A) \neq \emptyset$, and the position of the game defines a partial isomorphism. If the duplicator has a winning strategy that guarantees a win in k rounds, we write $(\mathfrak{A}, \bar{a}_0) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_0)$.

One can pass from the usual representation of games to the functional one without loss of generality:

Lemma 1. *Given $\mathfrak{F}' = \{\mathfrak{F}'(A, B)\}$, where each $\mathfrak{F}'(A, B)$ is a collection of families of relations on finite $A, B \subset U$, there is an agreement $\mathfrak{F}$ such that the relations $\equiv_k^{\mathfrak{F}'}$ and $\equiv_k^{\mathfrak{F}}$ coincide for every k.*

We now return to games seen in the previous example, and show how they look under the functional approach. We define three agreements: $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$, and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$.

- $\mathfrak{F}(\mathrm{FO})$: a tactic is a singleton set $\{f\}$, where $f : A \to B$ is a total function. Then $\mathfrak{F}(A, B)$, for each pair of finite sets (A, B), contains all possible tactics.
- $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$: same as above, except that each tactic is $\{f\}$ where $f : A \to B$ is a bijection (there are no tactics if $|A| \neq |B|$).
- $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$: given $A, B \subset U$, a tactic is a set $\mathcal{F}$ of partial maps such that for every $D \subseteq B$, there exists $f \in \mathcal{F}$ such that $\mathrm{dom}(f) = A$ and $|\{c \in A \mid f(c) \in D\}| \equiv |D| \pmod{p}$.

Definition 2. *Given an agreement $\mathfrak{F}$, we say that the $\mathfrak{F}$-game is a* game for a logic $\mathcal{L}$, *if there exists a partition $\{\mathcal{L}_0, \mathcal{L}_1, \dots\}$ of the formulae in $\mathcal{L}$ such that for every $k \geq 0$, there exists $k' \geq 0$ with the property that*

$$(\mathfrak{A}, \bar{a}_0) \equiv_{k'}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_0) \quad \textit{implies} \quad \big(\mathfrak{A} \models \varphi(\bar{a}_0) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b}_0)\big), \quad \textit{for all } \varphi \in \mathcal{L}_k.$$

If the converse holds as well, that is, for every $k' \geq 0$ there exists $k \geq 0$ such that, $(\mathfrak{A}, \bar{a}_0) \equiv_{k'}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_0)$, whenever $\mathfrak{A} \models \varphi(\bar{a}_0) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b}_0)$ for every $\varphi \in \mathcal{L}_k$, then we say that $\mathfrak{F}$-games capture *$\mathcal{L}$.*

Games are usually applied to prove *inexpressibility* results, in which case one only needs the condition that a given game is a game for a logic. In many cases, however, the converse holds too, that is, games completely characterize logics. The following is a reformulation, under our view of games, of standard results on characterizing logics by games [5, 17, 14, 16, 24, 28].

Proposition 1. *If $\mathcal{L}$ is one of* FO, $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$, *and* $\mathrm{FO}(\mathbf{Q}_p)$, *then $\mathfrak{F}(\mathcal{L})$-games are games for $\mathcal{L}$, with $\mathcal{L}_k$ being the set of $\mathcal{L}$-formulae of quantifier rank $\leq k$. Furthermore, these games capture the corresponding logic.*

4 Locality

Given a σ-structure $\mathfrak{A}$, its *Gaifman graph*, denoted by $G(\mathfrak{A})$, has A as the set of nodes. There is an edge (a_1, a_2) in $G(\mathfrak{A})$ iff there is a relation symbol R in σ

such that for some tuple t in the interpretation of this relation in $\mathfrak{A}$, both a_1, a_2 occur in t. By the *distance* $d(a_1, a_2)$ we mean the distance in the Gaifman graph, with $d(a, a) = 0$. If there is no path from a_1 to a_2 in $G(\mathfrak{A})$, then $d(a_1, a_2) = \infty$. We write $d(\bar{a}, b)$ for the minimum of $d(a, b)$ over a from $\bar{a}$.

Let $\mathfrak{A}$ be a σ-structure, and $\bar{a} = (a_1, \ldots, a_m) \in A^m$. The *radius r ball around $\bar{a}$* is the set $B_r^{\mathfrak{A}}(\bar{a}) = \{b \in A \mid d(\bar{a}, b) \leq r\}$. The *$r$-neighborhood of $\bar{a} = (a_1, \ldots, a_m)$ in $\mathfrak{A}$* is the structure $N_r^{\mathfrak{A}}(\bar{a})$ of vocabulary σ expanded with n constant symbols, where the universe is $B_r^{\mathfrak{A}}(\bar{a})$; σ-relations are restrictions of σ-relations in $\mathfrak{A}$ to $B_r^{\mathfrak{A}}(\bar{a})$, and the n additional constants are $a_1, \ldots, a_n$.

Since we define a neighborhood around an m-tuple as a structure with additional constant symbols, for any isomorphism h between $N_r^{\mathfrak{A}}(a_1, \ldots, a_m)$ and $N_r^{\mathfrak{B}}(b_1, \ldots, b_m)$, it must be the case that $h(a_i) = b_i$, $1 \leq i \leq m$.

Let $\mathfrak{A}, \mathfrak{B}$ be σ-structures, where σ only contains relation symbols. Let $\bar{a} \in A^m$ and $\bar{b} \in B^m$. We write $(\mathfrak{A}, \bar{a}) \leftrightarrows_d (\mathfrak{B}, \bar{b})$ if there exists a bijection $f : A \to B$ such that $N_d^{\mathfrak{A}}(\bar{a}c)$ and $N_d^{\mathfrak{B}}(\bar{b}f(c))$ are isomorphic, for every $c \in A$. This definition is most commonly used when $m = 0$; then $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ means that for some bijection $f : A \to B$, $N_d^{\mathfrak{A}}(c) \cong N_d^{\mathfrak{B}}(f(c))$ for all $c \in A$. That is, $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ realize the same multiset of isomorphism types of d-neighborhoods of points.

We say that a formula $\varphi(\bar{x})$ is *Hanf-local*, if there exists a number $d \geq 0$ such that $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b})$ whenever $(\mathfrak{A}, \bar{a}) \leftrightarrows_d (\mathfrak{B}, \bar{b})$. This concept was first introduced by Hanf [13] for FO over infinite structures, then modified by [8] to work for sentences over finite models.

Gaifman's theorem [10] states that every FO formula $\varphi(\bar{x})$ is equivalent to a Boolean combination of sentences and formulae in which quantification is restricted to $B_r(\bar{x})$, with r determined by φ. In particular, this implies that for every FO formula, we have two numbers, d and k, such that if $\mathfrak{A}$ and $\mathfrak{B}$ agree on all FO sentences of quantifier-rank $\leq k$ and $N_d^{\mathfrak{A}}(\bar{a}) \cong N_d^{\mathfrak{B}}(\bar{b})$, then $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b})$. This concept is normally used when $\mathfrak{A} = \mathfrak{B}$; then it says that a formula $\varphi(\bar{x})$ is *Gaifman-local* if there exists a number $d \geq 0$ such that for every structure $\mathfrak{A}$, if $N_d^{\mathfrak{A}}(\bar{a}_1) \cong N_d^{\mathfrak{A}}(\bar{a}_2)$, then $\mathfrak{A} \models \varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$.

A formula $\varphi(\bar{x})$ is *weakly-local* [21] if the above condition holds for disjoint neighborhoods: that is, there is a number $d \geq 0$ such that for every structure $\mathfrak{A}$, if $N_d^{\mathfrak{A}}(\bar{a}_1) \cong N_d^{\mathfrak{A}}(\bar{a}_2)$ and $B_d^{\mathfrak{A}}(\bar{a}_1) \cap B_d^{\mathfrak{A}}(\bar{a}_2) = \emptyset$, then $\mathfrak{A} \models \varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$.

The following implications are known [15, 21]: Hanf-local $\Rightarrow$ Gaifman-local $\Rightarrow$ weakly-local. Examples of logics in which all formulae are Hanf- (and hence Gaifman and weakly) local are all the logics considered so far: FO, FO($\mathbf{Q}_p$), $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ [10, 15, 20, 23]. There are examples of formulae that are Gaifman- but not Hanf-local [15] and weakly but not Gaifman-local [21].

We now state the definition that relaxes the concept of locality, by placing requirements weaker than isomorphism of neighborhoods. For $d, l \geq 0$, we use the notation $(\mathfrak{A}, \bar{a}) \leftrightarrows_{d,l}^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$ if there exists a bijection $f : A \to B$ such that $N_d^{\mathfrak{A}}(\bar{a}c) \equiv_l^{\mathfrak{F}} N_d^{\mathfrak{B}}(\bar{b}f(c))$ for every $c \in A$.

Definition 3. *An agreement* $\mathfrak{F}$ *is* Hanf-local *if for every* $k, m \in \mathbb{N}$*, there exists* $d, l \in \mathbb{N}$ *such that for every two structures* $\mathfrak{A}, \mathfrak{B}$*,* $\bar{a} \in A^m$ *and* $\bar{b} \in B^m$*,*

$$(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{d,l} (\mathfrak{B}, \bar{b}) \quad \Rightarrow \quad (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b}).$$

We call $\mathfrak{F}$ Gaifman-local *if for every* $k, m \in \mathbb{N}$*, there exists* $d, l \in \mathbb{N}$ *such that for every two structures* $\mathfrak{A}, \mathfrak{B}$*,* $\bar{a} \in A^m$ *and* $\bar{b} \in B^m$*,*

$$\mathfrak{A} \equiv^{\mathfrak{F}}_l \mathfrak{B} \quad \text{and} \quad N^{\mathfrak{A}}_d(\bar{a}) \equiv^{\mathfrak{F}}_l N^{\mathfrak{B}}_d(\bar{b}) \quad \Rightarrow \quad (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b}).$$

Finally, we call $\mathfrak{F}$ weakly-local *if for every* $k, m \in \mathbb{N}$*, there exist* $d, l \in \mathbb{N}$ *such that for every structure* $\mathfrak{A}$*,* $\bar{a} \in A^m$ *and* $\bar{b} \in A^m$*,*

$$N^{\mathfrak{A}}_d(\bar{a}) \equiv^{\mathfrak{F}}_l N^{\mathfrak{A}}_d(\bar{b}) \quad \text{and} \quad B^{\mathfrak{A}}_d(\bar{a}) \cap B^{\mathfrak{A}}_d(\bar{b}) = \emptyset \quad \Rightarrow \quad (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{A}, \bar{b}).$$

Our main question is the following: *When is a logic local under its games?* Or, more precisely: suppose $\mathfrak{F}$-games are games for a logic $\mathcal{L}$; is $\mathfrak{F}$ Hanf-, Gaifman-, or weakly-local?

If a logic is local under its games, we need an assumption *weaker* than isomorphism in order to prove that formulae cannot distinguish some elements of a structure. Consider, for example, the case of Gaifman-locality, applied to one structure $\mathfrak{A}$. Normally, to derive $\varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$, we would need to assume that $N_d(\bar{a}_1) \cong N_d(\bar{a}_2)$ for some appropriate d. But suppose we know that φ comes from a logic Gaifman-local under $\mathfrak{F}$-games. If k is such that $(\mathfrak{A}, \bar{a}_1) \equiv^{\mathfrak{F}}_k (\mathfrak{A}, \bar{a}_2)$ implies $\varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$, then we find $d, l \in \mathbb{N}$ that ensure

$$N^{\mathfrak{A}}_d(\bar{a}_1) \equiv^{\mathfrak{F}}_l N^{\mathfrak{A}}_d(\bar{a}_2) \ \Rightarrow \ (\mathfrak{A}, \bar{a}_1) \equiv^{\mathfrak{F}}_k (\mathfrak{A}, \bar{a}_2) \ \Rightarrow \ \mathfrak{A} \models \varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2).$$

Thus, instead of isomorphism of neighborhoods, we have a weaker requirement that they be indistinguishable by the $\mathfrak{F}$-game, in l rounds.

Even though the notion of locality under games is easier to apply, it is harder to analyze than the standard isomorphism-based locality. For example, if a logic $\mathcal{L}$ is local (Hanf-, or Gaifman-, or weakly) under isomorphisms, and $\mathcal{L}'$ is a sublogic of $\mathcal{L}$, then $\mathcal{L}'$ is local as well. The same, however, is *not* true for game-based locality, as we shall see, as properties of games guaranteeing locality need not be preserved if one passes to weaker games.

5 Basic structural properties

We now look at some most basic properties of agreements that are expected to hold. Intuitively, they are: (1) the spoiler is free to play any point he wants to; (2) the duplicator can mimic spoiler's moves when they play on the same structure; (3) the games on $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{B}, \mathfrak{C})$ can be composed into a single game on $(\mathfrak{A}, \mathfrak{C})$, and (4) agreements do not depend on a particular choice of elements of U.

Definition 4. *An agreement* $\mathfrak{F}$ *is said to be* admissible *if the following hold:*

(1) For every $\mathcal{F}(A,B) \in \mathfrak{F}$, we have $\bigcup\{\mathrm{dom}(f) \mid f \in \mathcal{F}(A,B)\} = A$ (the spoiler can play any point he wants to);
(2) For every $A \subset U$, there exists $\mathcal{F}(A,A) \in \mathfrak{F}$ such that every $f \in \mathcal{F}(A,A)$ is the identity on $\mathrm{dom}(f)$ (the duplicator can repeat spoiler's moves if they play on the same set);
(3) For every $\mathcal{F}(A,B)$, $\mathcal{F}(B,C) \in \mathfrak{F}$, the composition $\mathcal{F}(A,B) \circ \mathcal{F}(B,C) = \{g \circ f \mid f \in \mathcal{F}(A,B) \text{ and } g \in \mathcal{F}(B,C)\}$ is a tactic in $\mathfrak{F}$ (games compose);
(4) If $\mathcal{F}(A,B)$ is a tactic in $\mathfrak{F}$, and $g : A' \to A, h : B \to B'$ are bijections, then $\{h \circ f \circ g \mid f \in \mathcal{F}(A,B)\}$ is a tactic over A', B' (agreements do not depend on the choice of elements of U).

It is an easy observation that the agreements $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$, and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ are admissible.

Proposition 2. *Given an admissible agreement $\mathfrak{F}$ and $m, k \geq 0$,*

(a) $\equiv_k^{\mathfrak{F}}$ is an equivalence relation on structures $(\mathfrak{A}, \bar{a}), \bar{a} \in A^m$;
(b) If $h : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism, then $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, h(\bar{a}))$.

In many logics, the equivalence classes of $\equiv_k^{\mathfrak{F}}$ are definable by formulae (they correspond to *types*, or rank-k types, as k typically refers to the quantifier rank). Then definable sets are unions of types. We introduce an abstract notion of definable sets: a set $S \subseteq A^m$ is $(\mathfrak{F}, k)$-*definable* in $\mathfrak{A}$ if it is closed under $\equiv_k^{\mathfrak{F}}$: that is, $\bar{a} \in S$ and $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{A}, \bar{a}_1)$ imply $\bar{a}_1 \in S$. For admissible agreements, definable sets behave in the expected way.

Proposition 3. *If $\mathfrak{F}$ is an admissible agreement, then $(\mathfrak{F}, k)$-definable sets are closed under Boolean combinations and Cartesian product; furthermore, the projection $A^{m+1} \to A^m$ applied to an $(\mathfrak{F}, k)$-definable set produces an $(\mathfrak{F}, k+1)$-definable set.*

6 Weak locality

We now move to the first locality condition, weak locality. In many applications of locality, at least for proving expressibility bounds, one actually uses weak locality as it is easier to work with disjoint neighborhoods (see, e.g., Fig. 1). While examples of weakly-local formulae violating other notions of locality exist, they are not particularly natural [21].

To guarantee weak locality, we impose two very mild conditions on $\mathfrak{F}$-games. The first has to do with compositionality. Composition of games is a standard technique that allows one to use $\mathfrak{A} \equiv_k^{\mathfrak{F}} \mathfrak{A}'$ and $\mathfrak{B} \equiv_k^{\mathfrak{F}} \mathfrak{B}'$ to conclude $\mathcal{H}(\mathfrak{A}, \mathfrak{B}) \equiv_l^{\mathfrak{F}} \mathcal{H}(\mathfrak{A}', \mathfrak{B}')$, for some operation $\mathcal{H}$ (see, e.g., [22] for a survey). While in general such compositionality properties depend on the type of games and the operator $\mathcal{H}$, there is one scenario where they almost universally apply: when $\mathcal{H}$ is the disjoint union of structures [22] (in fact, l is usually equal to k in this situation). We want our games to satisfy this property. We use $\sqcup$ for disjoint union of sets and functions.

Definition 5. *An agreement $\mathfrak{F}$ is* compositional, *if for every two tactics $\mathcal{F}(A,B)$ and $\mathcal{G}(C,D)$ in $\mathfrak{F}$ such that $A \cap C = B \cap D = \emptyset$, the tactic $\mathcal{F}(A,B) \sqcup \mathcal{G}(C,D)$ defined as the set of disjoint unions of partial functions $f : A \to B$ from $\mathcal{F}(A,B)$ and $g : C \to D$ from $\mathcal{G}(C,D)$ is in $\mathfrak{F}$.*

The second condition says that if in a game $\mathfrak{A} \equiv_k^{\mathfrak{F}} \mathfrak{B}$, both players play restricted to subsets $C \subseteq A$ and $D \subseteq B$, then such a game may be considered as a game on substructures of $\mathfrak{A}$ and $\mathfrak{B}$ generated by C and D, respectively. Again, this condition is true for practically all reasonable games.

We formalize it as follows. We denote the set of all nonempty restrictions of partial functions from $\mathcal{F}(A,B)$ to $C \subseteq A$ by $\mathcal{F}(A,B)|_C$. Consider a tactic $\mathcal{F}(A,B)$, and nonempty sets $C \subseteq A$ and $D \subseteq B$. We say that $\mathcal{F}(A,B)$ is *shrinkable* to (C,D) if $a \in C \Leftrightarrow f(a) \in D$ for every $f \in \mathcal{F}(A,B)$ and $a \in \mathrm{dom}(f)$.

Definition 6. *An agreement $\mathfrak{F}$ is* shrinkable *if for every $\mathcal{F}(A,B) \in \mathfrak{F}$, and nonempty subsets $C \subseteq A$ and $D \subseteq B$, if $\mathcal{F}(A,B)$ is shrinkable to (C,D), then $\mathcal{F}(A,B)|_C$ is a tactic over (C,D) that belongs to $\mathfrak{F}$.*

An admissible $\mathfrak{F}$ is called basic *if it is both shrinkable and compositional.*

A simple examination of the agreements seen so far in this paper shows:

Proposition 4. *The agreements $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ are basic.*

Recall that an agreement $\mathfrak{F}$ is weakly-local if for every $k, m \geq 0$, there exist $d, l \geq 0$ such that for every structure $\mathfrak{A}$ and every $\bar{a}, \bar{b} \in A^m$, if $N_d^{\mathfrak{A}}(\bar{a}) \equiv_l^{\mathfrak{F}} N_d^{\mathfrak{A}}(\bar{b})$ and the neighborhoods $N_d^{\mathfrak{A}}(\bar{a})$ and $N_d^{\mathfrak{A}}(\bar{b})$ are disjoint, then $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{A}, \bar{b})$. We define the *weak-locality rank with respect to* $\mathfrak{F}$, denoted by $\mathrm{wlr}_{\mathfrak{F}}(k,m)$, as the minimum d for which the above condition holds.

Theorem 1. *Every basic agreement $\mathfrak{F}$ is weakly-local. Furthermore, $\mathrm{wlr}_{\mathfrak{F}}(k,m) = O(2^k)$.*

Corollary 1. *The agreements $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ are weakly-local.*

That is, FO, $\mathrm{FO}(\mathbf{Q}_p)$, and $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ are weakly-local under their games.

Nevertheless, there are extensions of FO with simple unary generalized quantifiers that are not weakly-local under their games.

Let PRIME be the set of primes and $\mathbf{Q}_{\mathrm{PRIME}}$ the corresponding generalized quantifier. That is, $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$ extends FO with formulae $\mathbf{Q}_{\mathrm{PRIME}} y\, \varphi(\bar{x}, y)$ such that $\mathfrak{A} \models \mathbf{Q}_{\mathrm{PRIME}} y\, \psi(\bar{a}, y)$ if $|\{a \mid \mathfrak{A} \models \psi(\bar{a}, a)\}|$ is a prime number. We show that $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$ is not weakly-local under its games.

We first define the agreement $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$. For two finite sets $A, B \subset U$, a tactic is a set $\mathcal{F}$ of partial maps such that for every nonempty $D \subseteq B$, there exists $f \in \mathcal{F}$ such that $\mathrm{dom}(f) = A$ and $|f^{-1}(D)| \in \mathrm{PRIME}$ iff $|D| \in \mathrm{PRIME}$. (In terms of the game, in every round the spoiler selects a set $D \subseteq B$; the duplicator selects $C \subseteq A$ such that $|C|$ is prime iff $|D|$ is. Then the spoiler plays $a \in A$ and the duplicator responds with $b \in B$ such that $a \in C$ iff $b \in D$.) Notice that this agreement is not compositional, and hence not basic.

It is known [28] that for every $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$-formula $\varphi(\bar{x})$ of quantifier rank k, if $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))} (\mathfrak{B}, \bar{b})$, then $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(\bar{b})$. Thus, to show that $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$ is not weakly-local under its games, it suffices to prove the following:

Proposition 5. $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$ *is not weakly-local.*

For this, we give a formula $\varphi(x)$ such that for every $d, l \geq 0$, there is a structure $\mathfrak{A}$ and $a, b \in A$ such that $N_d^{\mathfrak{A}}(a) \equiv_l^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))} N_d^{\mathfrak{A}}(b)$, $B_d^{\mathfrak{A}}(a) \cap B_d^{\mathfrak{A}}(b) = \emptyset$, and yet $\mathfrak{A} \models \varphi(a) \wedge \neg\varphi(b)$.

Let σ be a signature of a unary relation R and a binary relation E, and let $d, l \geq 0$. Consider the structure $\mathfrak{A}$ whose E-relation is shown in Fig. 2 below; the relation R is interpreted as the set of all a_i's, b_i's, and c_i's. Let $\varphi(x)$ be $\mathbf{Q}_{\mathrm{PRIME}} y\, (R(y) \wedge \neg E(x, y))$.

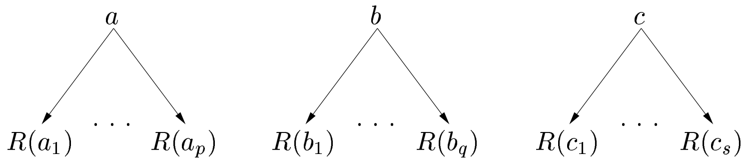

Fig. 2. A structure for proving that $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$ is not weakly-local under its games.

There are infinitely many primes r such that all the numbers $r - i$ $(i \leq l)$ are composite. Choose two sufficiently large p, q $(p \neq q)$ from this set so that $N_d^{\mathfrak{A}}(a) \equiv_l^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))} N_d^{\mathfrak{A}}(b)$ (notice that d can be taken to be 1, without loss of generality). By Dirichlet's Theorem, the arithmetic progression $np + q$ $(n = 0, 1, \ldots)$ contains an infinite number of primes. Let $n \geq 1$ be such that $np + q$ is a prime and let $s = np$. Then, $\mathfrak{A} \models \varphi(a)$, since $q + s = np + q$ is prime, and $\mathfrak{A} \not\models \varphi(b)$, since $p + s = (n+1)p$ is composite. Thus, the agreement $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$ is not weakly-local.

7 Hanf-locality

We now present a condition that guarantees Hanf-locality of agreements. While still easy to state, this condition already fails for some logics, notably for FO.

We say that $\mathcal{F}(A, B)$ is a *matching* tactic, if the union $\bigcup_{f \in \mathcal{F}(A,B)} \mathrm{graph}(f)$ is a matching on $A \times B$. That is, the union of all the functions from $\mathcal{F}(A, B)$ is a partial bijection. For example, all the tactics in $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ are matching.

From a tactic $\mathcal{F}(A, B)$ we define a relation $\approx_{\mathcal{F}(A,B)}$ as the minimal relation that contains $\bigcup_{f \in \mathcal{F}(A,B)} \mathrm{graph}(f)$ and satisfies the following: if $a \approx_{\mathcal{F}(A,B)} b'$, $a' \approx_{\mathcal{F}(A,B)} b$ and $f(a') = b'$ for some $f \in \mathcal{F}(A, B)$, then $a \approx_{\mathcal{F}(A,B)} b$.

Another way of looking at this relation is the following: $a \approx_{\mathcal{F}(A,B)} b$ if there is a sequence $\langle a_0, b_1, a_1, b_2, a_2, \ldots, b_{m-1}, a_{m-1}, b_m \rangle$ where $a_0 = a$, $b_m = b$, and for every i, there are $f, f' \in \mathcal{F}(A,B)$ such that $b_i = f(a_{i-1}) = f'(a_i)$, $1 \leq i \leq m-1$, and $b_m = f(a_{m-1})$ for some $f \in \mathcal{F}(A,B)$.

Definition 7. *An agreement $\mathfrak{F}$ is called* matching *if for every tactic $\mathcal{F}(A,B) \in \mathfrak{F}$, there exists a matching tactic $\mathcal{G}(A,B) \in \mathfrak{F}$ such that $\bigcup_{g \in \mathcal{G}(A,B)} \mathrm{graph}(g)$ is contained in $\approx_{\mathcal{F}(A,B)}$.*

If every tactic in an agreement is matching, then the agreement itself is matching. However, some agreements can be matching and have non-matching tactics (examples will be given in the full version of the paper). The following holds trivially:

Proposition 6. *$\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ is a matching agreement.*

Recall that an agreement $\mathfrak{F}$ is Hanf-local, if for every $k, m \geq 0$ there exist $d, l \geq 0$ such that, for every two structures $\mathfrak{A}$, $\mathfrak{B}$ and every $\bar{a} \in A^m$ and $\bar{b} \in B^m$, if $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{d,l} (\mathfrak{B}, \bar{b})$, then $(\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b})$. The minimum d for which the above condition holds is called the *Hanf-locality rank with respect to* $\mathfrak{F}$, and is denoted by $\mathrm{hlr}_{\mathfrak{F}}(k,m)$.

Theorem 2. *If an agreement $\mathfrak{F}$ is basic and matching, then it is Hanf-local. Furthermore,* $\mathrm{hlr}_{\mathfrak{F}}(k,m) = O(2^k)$.

Corollary 2. *$\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ is Hanf-local, and* $\mathrm{hlr}_{\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))}(k,m) = O(2^k)$.

Thus, $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ is Hanf-local under its games. This nice behavior, however, does not extend to other logics known to possess the isomorphism-based Hanf-locality property.

Proposition 7. (see [25]) FO *and* $\mathrm{FO}(\mathbf{Q}_p)$ *are* not *Hanf-local under their games.*

For FO, this is proved by taking G_1 to be the complete graph with $2N$ vertices, and G_2 to be the disjoint union of two complete graphs with N vertices each. For every d and l, any bijection between the nodes of these graphs witnesses $G_1 \leftrightarrows^{\mathfrak{F}(\mathrm{FO})}_{d,l} G_2$ as long as $N > l$, and yet G_1 and G_2 disagree on $\exists x \exists y \neg E(x,y)$. For $\mathrm{FO}(\mathbf{Q}_p)$, the same proof works, but N is taken to be $p \cdot (l+1)$.

8 Gaifman-locality

Recall that $\mathfrak{F}$ is Gaifman-local if for every $k, m \geq 0$ there exist $d, l \geq 0$ such that, for every $\mathfrak{A}$ and $\mathfrak{B}$ and every $\bar{a} \in A^m$ and $\bar{b} \in B^m$, we have $(\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b})$ whenever $\mathfrak{A} \equiv^{\mathfrak{F}}_l \mathfrak{B}$ and $N^{\mathfrak{A}}_d(\bar{a}) \equiv^{\mathfrak{F}}_l N^{\mathfrak{B}}_d(\bar{b})$. The minimum such d is called *Gaifman-locality rank with respect to* $\mathfrak{F}$, and denoted by $\mathrm{lr}_{\mathfrak{F}}(k,m)$,

Our goal is to show that agreements defining games for FO, $\mathrm{FO}(\mathbf{Q}_p)$, and $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$, are all Gaifman-local. The proof of this fact is easier for more expressive logics such as $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ (this will be explained shortly). In that case, one can show the following:

Lemma 2. *If $\mathfrak{F}$ is a basic and matching agreement, then $\mathfrak{F}$ is Gaifman-local, and* $\mathrm{lr}_{\mathfrak{F}}(k,m) \leq 3 \cdot \mathrm{hlr}_{\mathfrak{F}}(k,m) + 1$.

This tells us that $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ is Gaifman-local under their games:

Corollary 3. *$\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ is Gaifman-local, and* $\mathrm{lr}_{\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))}(k,m) = O(2^k)$.

We next move to Gaifman-locality for FO and FO with modulo quantifiers. Gaifman-locality for them is the hardest of the locality conditions we consider here, mainly because of the following three reasons. First, it requires reasoning about overlapping neighborhoods, which is known to cause complications in the study of locality (see, e.g., [11]). Second, it is a strong notion that implies the existence of normal forms for logical formulae. Such normal forms have been shown for FO [10, 25]. Third, while establishing Gaifman-locality and normal forms, we match the best bound for Gaifman-locality rank for FO, $O(4^k)$. (In Gaifman's original proof, it was $O(7^k)$, the $O(4^k)$ bound is from [18]. For the "one-structure" version, and the isomorphism-based locality, the bound can be further reduced to $O(2^k)$ [20].)

We now show that logics which are Gaifman-local under their games admit a normal form, under the condition that the relations $\equiv_k^{\mathfrak{F}}$ are of finite index (as they are for FO and several other logics). In that case, every formula is equivalent to a Boolean combination of sentences and formulae evaluated in a neighborhood of its free variables. More precisely, for a logic $\mathcal{L}$ that satisfies the basic closure properties of [4] (that is, any reasonable logic, e.g., closed under $\vee, \wedge, \neg$), we can show the following.

Theorem 3. *Let $\mathcal{L}$ be a logic captured by an admissible Gaifman-local agreement $\mathfrak{F}$, where $\mathfrak{F}$ has the property that for every k, the relations $\equiv_k^{\mathfrak{F}}$ are of finite index. Then, for every $\mathcal{L}$-formula $\varphi(\bar{x})$, one can find a number d, a sequence $\Phi_1, \ldots, \Phi_n$ of $\mathcal{L}$-sentences, a sequence $\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})$ of $\mathcal{L}$-formulae, and a Boolean function $\beta : \{0,1\}^{n+m} \to \{0,1\}$ such that*

$$\mathfrak{A} \models \varphi(\bar{a}) \quad \Leftrightarrow \quad \beta\Big(\Phi_1(\mathfrak{A}), \ldots, \Phi_n(\mathfrak{A}), \varphi_1(N_d^{\mathfrak{A}}(\bar{a})), \ldots, \varphi_m(N_d^{\mathfrak{A}}(\bar{a}))\Big) = 1$$

where

$$\Phi_i(\mathfrak{A}) = \begin{cases} 1 & \text{if } \mathfrak{A} \models \Phi \\ 0 & \text{if } \mathfrak{A} \models \neg\Phi \end{cases} \qquad \text{and} \qquad \varphi_j(N_d^{\mathfrak{A}}(\bar{a})) = \begin{cases} 1 & \text{if } N_d^{\mathfrak{A}}(\bar{a}) \models \varphi_j(\bar{a}) \\ 0 & \text{if } N_d^{\mathfrak{A}}(\bar{a}) \models \neg\varphi_j(\bar{a}). \end{cases}$$

Thus, proving Gaifman-locality under games is comparable to proving a result like Gaifman's theorem itself. We now do this for FO and the following generalization of $\mathrm{FO}(\mathbf{Q}_p)$.

If $p_1, \ldots, p_r$ is a sequence of numbers, then $\mathrm{FO}(\mathbf{Q}_{p_1}, \ldots, \mathbf{Q}_{p_r})$ extends FO with all the generalized quantifiers $\mathbf{Q}_{p_i}$'s. This logic is captured by $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{p_1}, \ldots, \mathbf{Q}_{p_r}))$-games, where each tactic in the agreement $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{p_1}, \ldots, \mathbf{Q}_{p_r}))$ is simply a union of tactics from each of the $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{p_i}))$'s.

Theorem 4. *If $\mathfrak{F}$ is either $\mathfrak{F}(\mathrm{FO})$ or $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{p_1},\ldots,\mathbf{Q}_{p_r}))$, for an arbitrary sequence $p_1,\ldots,p_r$, then $\mathfrak{F}$ is Gaifman-local. Furthermore,* $\mathrm{lr}_{\mathfrak{F}}(k,m) = O(4^k)$.

Note that the bound shown for both FO and $\mathrm{FO}(\mathbf{Q}_{p_1},\ldots,\mathbf{Q}_{p_r})$ matches the best bound previously known for FO [18]. Furthermore, since for both FO and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{p_1},\ldots,\mathbf{Q}_{p_r}))$ the relation $\equiv_k^{\mathfrak{F}}$ is of finite index, the normal form result (Theorem 3) applies to them. For FO, this is of course known and follows from local normal forms of [10, 25]. Our proof, however, is new, and is based entirely on structural properties of games.

9 Conclusions

We looked at the natural extensions of three standard locality notions that use logical equivalence (or equivalence under games) of neighborhoods, as opposed to a much stronger condition of isomorphisms. Such locality notions can be applied in several scenarios where the standard isomorphism-based notions of locality are inapplicable. In fact, their applicability to FO has already been used in data exchange and integration scenarios to help draw the boundary between rewritable and non-rewritable queries [1].

We defined an abstract view of games that let us consider the notions of locality in an abstract setting, independent of a particular logic. This approach is applicable to many logics which are captured by games and whose types are definable in the logic itself (with some exceptions, of course, such as finite variable logics [3], but some of them are non-local). We identified conditions that guarantee the main notions of locality for those games.

The notions for which most questions remain is Gaifman-locality. Unlike others, which admit $O(2^k)$ bounds on locality rank, for Gaifman-locality we could only show a $O(4^k)$ bound, and even that matches the very recently discovered bound for FO, as those previously known were of the order of 7^k. We would like to settle the case of Gaifman-locality completely, by finding natural conditions for it that account for all the known cases, and by precisely calculating the locality rank.

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