n-distributivity, dimension and Carathéodory's theorem

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Abstract

A. Huhn proved that the dimension of Euclidean spaces can be characterized through algebraic properties of the lattices of convex sets. In fact, the lattice of convex sets of $\mathbb{E}^n$ is $n+1$-distributive but not n-distributive . In this paper his result is generalized for a class of algebraic lattices generated by their completely join-irreducible elements. The lattice theoretic form of Carathéodory's theorem characterizes n-distributivity in such lattices. Several consequences of this result are studied. First, it is shown how infinite n-distributivity and Carathéodory's theorem are related. Then the main result is applied to prove that for a large class of lattices being n-distributive means being in a variety generated by the finite n-distributive lattices. Finally, n-distributivity is studied for various classes of lattices, with particular attention being paid to convexity lattices of Birkhoff and Bennett for which a Helly type result is also proved.

1 Introduction

It was discovered recently that the dimension of Euclidean spaces (more generally, of vector spaces over ordered division rings) has a lattice theoretic characterization. There were two approaches to the problem, both getting dimension as an algebraic property of lattices of convex sets. Huhn [13] studied the lattice of convex sets of n-dimensional Euclidean space $\mathbb{E}^n$. He observed that dimension can be characterized via *n-distributivity.* A lattice L is called *n-distributive* [12] if, for any $x, y_0, \ldots, y_n$, the following equation holds:

$$(\mathbf{D}_n) \qquad x \wedge \bigvee_{i=0}^{n} y_i = \bigvee_{i=0}^{n} (x \wedge \bigvee_{j \neq i} y_j)$$

Huhn proved that the lattice of convex sets of $\mathbb{E}^n$, denoted by $Co(\mathbb{E}^n)$, is $n+1$-distributive but is not n-distributive. The main tool to prove this result was Carathéodory's theorem saying that in $\mathbb{E}^n$, if a point is in the convex hull of $m > n+1$ points, then it is in the convex hull of at most $n+1$ of those points [27]. Moreover, it was shown that the dual of $Co(\mathbb{E}^n)$ is $n+1$-distributive but is not n-distributive. This fact was derived from Helly's theorem saying that in $\mathbb{E}^n$, a finite family of convex sets has a nonempty intersection whenever any $n+1$ sets have a non-empty intersection [27].

In [4] Birkhoff and Bennett introduced convexity lattices which arise naturally when one studies a ternary relation of betweenness β, $(xyz)\beta$ meaning y lies between x and z, and the lattice of convex sets with respect to this relation. A set X is called convex if $x, z \in X$ and $(xyz)\beta$ imply $y \in X$.

Several restrictions reminiscent of Hilbert's connection and order axioms were imposed. The modular core of a convexity lattice was interpreted as the lattice of affine flats which was shown to be a geometric lattice under certain conditions. Its height (to be more precise, height minus one) was interpreted as the dimension. Of course, if β is the usual betweenness in $\mathbb{E}^{\mathbb{K}}$, such defined dimension of $Co(\mathbb{E}^{\mathbb{K}})$ is n. It was proved in [4] that lattice theoretic versions of theorems of Radon, Helly and Carathéodory determine the dimension.

The two approaches are not unrelated at all. In fact, one can easily rewrite the proof of [13] to show that if Carathéodory's theorem of dimension n holds in a convexity lattice (which means its dimension defined as the height of the modular core is n [4]) then this convexity lattice is indeed $n+1$-distributive but not n-distributive. However, being isomorphic to $Co(\mathbb{E}^{\mathbb{K}})$ or even being a convexity lattice is too much of an assumption to prove that Carathéodory's theorem and n-distributivity are related. Convexity lattices (of which $Co(\mathbb{E}^{\mathbb{K}})$ is an example) enjoy some nice algebraic properties. In particular, they are algebraic atomistic lattices. We will show that being algebraic and atomistic is enough to prove the intimate connection between n-distributivity and the lattice-theoretic version of Carathéodory's theorem. In fact, even this is too strong: all that is needed is algebraicity and the assumption that every element of a lattice is the join of completely join-irreducible elements below it.

n-distributivity can be viewed as a notion weaker than distributivity: $\mathbf{D}_n$ implies $\mathbf{D}_m$ if $n < m$ and $\mathbf{D}_1$ is the usual distributivity. It is well-known that algebraic distributive lattices satisfy the law of infinite join-distributivity: $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$ [14]. Complete lattices satisfying this law are called frames. They may arise as lattices of open sets of topological spaces. It was shown in [15] that the ideal completion is left adjoint to the forgetful functor from the category of frames to the category of distributive lattices[1]. Furthermore, a certain subcategory of the category frames which corresponds to so-called coherent spaces turns out to be equivalent to the category of distributive lattices. We shall use the main characterization theorem to extend these results to n-distributivity. Every algebraic n-distributive lattice satisfies the *infinite n-distributive* law:

$$(\mathbf{ID}_n) \qquad x \wedge \bigvee_{i\in I} y_i = \bigvee_{K\subseteq I, |K|=n} (x \wedge \bigvee_{j\in K} y_j)$$

It will be shown that the ideal completion is left adjoint to the forgetful functor from $\mathbf{ID}_n$ to $\mathbf{D}_n$ considered as categories. To find an analog of the second fact mentioned above, we consider convexities rather than topologies. There is a notion of an *(abstract) convexity* [30, 32, 10] and the abstract (or axiomatic) theory of convex spaces is well-developed. In this paper we define what it means for an abstract convexity to be n-dimensional. Having defined it, we show that n-dimensional convexities can be given the structure of a category which is equivalent to a certain full subcategory of the category of $n+1$-distributive lattices.

So much for categories, let's turn to varieties. Let Δ_n be the variety of n-distributive lattices and $\Delta_n^{\mathcal{F}}$ the minimal variety that contains all finite n-distributive lattices, i.e. $\mathbf{HSP}(\Delta_n \cap \mathcal{F})$ where $\mathcal{F}$ is the class of finite lattices. It was proved in [13] that $Co(\mathbb{E}^{\mathbb{K}})$ is in $\Delta_n^{\mathcal{F}}$ and that $\mathsf{M} \cap \Delta_n = \mathsf{M} \cap \Delta_n^{\mathcal{F}}$ where M is the variety of modular lattices. In this paper we generalize these results in two ways using our main characterization of n-distributivity . First, any algebraic lattice in which every element is the join of completely join-irreducible elements is in Δ_n iff it is in $\Delta_n^{\mathcal{F}}$, hence the first result. Furthermore, if a variety $\mathcal{V}$ is such that any lattice $L \in \mathcal{V}$ can be embedded into $L' \in \mathcal{V}$ such that L' is algebraic, every element of L' is the join of completely join-irreducible elements and the embedding preserves

[1] Of course the algebraic members of a variety always form a (non-full) reflective subcategory, but not all frames are algebraic.

identities, then $\mathcal{V} \cap \Delta_n = \mathcal{V} \cap \Delta_n^{\mathcal{F}}$. Since M is such, we obtain the second result.

Our characterization of n-distributivity via the Carathéodory condition can be applied to obtain nice characterizations of n-distributivity in several classes of lattices. For example, in geometric lattices n-distributivity is related to the sizes of circuits of underlying matroids. As a by-product of our study of n-distributivity in planar lattices we show that any lattice of the order-theoretic dimension n is n-distributive.

Having forgotten about convexity lattices for a while, we return to them in the last section. It is shown that a convexity lattice of dimension n is what we call "an abstract convexity of dimension n" which is defined in terms of n-distributivity when we establish the equivalence of categories. Then we use Helly's theorem for convexity lattices to show that their dimensions can be defined via the dual n-distributivity as well.

In the rest of this section we give all necessary definitions (cf. [6, 14]). The rest of the paper is organized in five sections. In Section 2 we prove the main theorem stating that an algebraic lattice in which every element is the join of completely join-irreducible elements is n-distributive iff Carathéodory's theorem of dimension $n-1$ holds. Using this result, we prove a characterization theorem for the infinite n-distributivity and establish the equivalence of categories of what we call convexities of dimension $n-1$ and certain n-distributive lattices. In Section 3 the results about varieties Δ_n and $\Delta_n^{\mathcal{F}}$ are proved. In Section 4 we consider examples. Section 5 deals with convexity lattices. Concluding remarks are given in Section 6.

Let L be a complete lattice. An element x of L is called *completely join-irreducible* if $x = \bigvee X$ implies $x \in X$. The set of all completely join-irreducible elements is denoted by $CJ(L)$. A complete lattice L is called *CJ-generated* if $x = \bigvee CJ(x)$ where $CJ(x) = {\downarrow} x \cap CJ(L)$ (they were called V_1-lattices in [29]). An element x is called *complete prime* if $x \leq \bigvee X$ implies $x \leq x'$ for some x' in X and *n-complete prime* if $x \leq \bigvee X$ implies that there are n elements $x_1, \ldots, x_n \in X$ such that $x \leq x_1 \vee \ldots \vee x_n$. The set of n-complete primes is denoted by $CP_n(L)$.

A complete lattice is called *atomistic* if every element in it is the join of atoms. Atomistic lattices are obviously CJ-generated. The lattice $Co(\mathbb{E}^{\ltimes - \nVdash})$ is atomistic and Carathéodory's theorem has the following lattice theoretic formulation: Given atoms $a, b_1, \ldots, b_m \in Co(\mathbb{E}^{\ltimes - \nVdash})$ such that $a \leq b_1 \vee \ldots \vee b_m$ and $m > n$, there exist n indices $i_1, \ldots, i_n$ in $\{1, \ldots, m\}$ such that $a \leq b_{i_1} \vee \ldots b_{i_n}$. We use $n-1$-dimesnional space here because the least k such that the lattice of convex sets becomes k-distributive is the dimension plus one.

Motivated by this, we give the following definition. A CJ-generated lattice L is said to satisfy *the Carathéodory condition of dimension* $n-1$, or (cc_n) for short, if the following holds:

If $a, b_1, \ldots, b_m \in CJ(L)$, $a \leq b_1 \vee \ldots \vee b_m$ and $m > n$, then there exist n indices $i_1, \ldots, i_n$ in $\{1, \ldots, m\}$ such that $a \leq b_{i_1} \vee \ldots \vee b_{i_n}$

The Carathéodory rank of a lattice is the minimal n such that (cc_n) holds. If no such n exists, the rank is ∞. Similarly, the Huhn rank of a lattice is the minimal n such that the lattice is n-distributive. If no such n exists, the rank is ∞. Both Carathéodory and Huhn ranks of $Co(\mathbb{E}^{\ltimes - \nVdash})$ are n.

2 n-distributivity and the Carathéodory condition

In this section we prove our main result stating that for algebraic CJ-generated lattices the Carathéodory rank equals the Huhn rank. We also study infinite n-distributivity in such lattices and discover an equivalence between a subcategory of $n+1$-distributive lattices and the category of what we call convexities of dimension n. It is also shown how n-distributivity and closure ranks [23] are related.

Theorem 2.1 *Given an algebraic CJ-generated lattice L, the following are equivalent:*
1) L is n-distributive;
2) L is infinitely n-distributive;
3) (cc_n) holds in L;
4) $CJ(L) \subseteq CP_n(L)$.

Proof: The most important part of the proof is the equivalence 1) $\Leftrightarrow$ 3). Any infinitely n-distributive lattice is always n-distributive, so to prove 1) $\Leftrightarrow$ 2) it is enough to show that an algebraic n-distributive lattice is infinitely n-distributive. The remaining equivalence 3) $\Leftrightarrow$ 4) is rather straightforward.

1) $\Rightarrow$ 3). Assume that L is n-distributive but (cc_n) does not hold. That means, there exist $a, b_1, \ldots, b_m$ in $CJ(L)$ such that $a \leq b_1 \vee \ldots b_m$ but $a \not\leq b_{i_1} \vee \ldots \vee b_{i_n}$ for every sequence of n indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$. Then there exists a number p such that $n \leq p < m$ and $a \not\leq b_{i_1} \vee \ldots \vee b_{i_p}$ for every sequence of p indices $i_1, \ldots, i_p \in \{1, \ldots, m\}$ and p is maximal such. That means, $a \leq b_{i_1} \vee \ldots \vee b_{i_{p+1}}$ for some choice of $p+1$ indices. Since $n \leq p$, L is p-distributive. Therefore,

$$a = a \wedge \bigvee_{j=1}^{p+1} b_{i_j} = \bigvee_{j=1}^{p+1} (a \wedge \bigvee_{l \neq j} b_{i_l})$$

Since a is join-irreducible, $a \leq \bigvee_{l \neq j} b_{i_l}$ for some j, which means a is under the join of p elements from $\{b_1, \ldots, b_m\}$, a contradiction.

3) $\Rightarrow$ 1). Let L be an algebraic CJ-generated lattice satisfying (cc_n). Prove that L is n-distributive. The $\geq$ inequality always holds for the left and right hand sides of $\mathbf{D}_n$. Since L is CJ-generated, it is therefore enough to prove that for any $a \in CJ(L)$, $a \leq x \wedge \bigvee_{i=0}^{n} y_i$ implies $a \leq \bigvee_{i=0}^{n}(x \wedge \bigvee_{j \neq i} y_j)$. Let $Y = CJ(y_0) \cup \ldots \cup CJ(y_n)$. Then $a \leq \bigvee_{i=0}^{n} y_i = \bigvee Y$. Since a is compact, there is a finite subset $\{z_0, \ldots, z_p\}$ of Y such that $a \leq z_0 \vee \ldots \vee z_p$. If $p \geq n$, by (cc_n) there exist n indices $i_1, \ldots, i_n$ such that $a \leq z_{i_1} \vee \ldots \vee z_{i_n}$ since a and all z_i's are in $CJ(L)$. Therefore, we may assume without loss of generality that $p < n$. Then each z_l is under some y_{i_l} and a is below the join of at most n y_j's. Hence, $a \leq x \wedge \bigvee_{j \neq i} y_j$ for some i. 3) $\Rightarrow$ 1) is proved.

Every algebraic n-distributive lattice L is infinitely n-distributive. Again, the left hand side of $\mathbf{ID}_n$ is always greater than the right hand side. To prove our claim we must show that any compact $a \leq x \wedge \bigvee_{i \in I} y_i$ is also below

$$\bigvee_{K \subseteq I, |K| = n} (x \wedge \bigvee_{j \in K} y_j)$$

Since a is compact and $a \leq \bigvee_{i \in I} y_i$, $a \leq y_{i_1} \vee \ldots \vee y_{i_p}$ for finitely many $i_1, \ldots, i_p \in I$. If $p \leq n$, we are done. If $p > n$, applying $m-$distributivity of L for $m = p, p-1, \ldots, n$ yields

$$a \leq \bigvee_{j=1}^{p} (x \wedge \bigvee_{l \neq j} y_{i_l}) \quad = \bigvee_{K \subseteq \{1, \ldots, p\}, |K| = n} (x \wedge \bigvee_{l \in K} y_{i_l}) \ \leq \bigvee_{K \subseteq I, |K| = n} (x \wedge \bigvee_{j \in K} y_j).$$

This finishes the proof of infinite n-distributivity of L and the theorem. □

Corollary 2.2 *For any algebraic CJ-generated lattice, its Carathéodory and Huhn ranks coincide.* □

Our next goal is to characterize infinite n-distributivity in CJ-generated lattices via the Carathéodory condition.

Corollary 2.3 *A CJ-generated lattice is infinitely n-distributive iff it is algebraic and* (cc_n) *holds.*

Proof: If L is CJ-generated, algebraic and (cc_n) holds, then L is infinitely n-distributive by theorem 2.1. Conversely, let L be infinitely n-distributive CJ-generated lattice. Show that L is algebraic. Let $a \in CJ(L)$ and $a \leq \bigvee X$. $\mathbf{ID}_n$ implies

$$a = a \wedge \bigvee X \quad = \bigvee_{X_f \subseteq X, |X_f| = n} (a \wedge \bigvee X_f)$$

Since $a \in CJ(L)$, there exists $X_f \subseteq X$ such that $|X_f| = n$ and $a = a \wedge \bigvee X_f$, i.e. $a \leq \bigvee X_f$. Thus, a is compact and L is algebraic. Corollary is proved. □

Corollary 2.3 shows that algebraicity can not be dropped if we want to prove that 2) and 3) of theorem 2.1 are equivalent. However, the question whether algebraicity is needed is justified if we are concerned with the equivalence of n-distributivity and (cc_n). It was proved in [13] that the lattice of *closed* convex sets of $\mathbb{E}^{\kappa}$ is $n+1$-distributive. The Carathéodory condition of dimension n (i.e. (cc_{n+1})) is true in that lattice but algebraicity fails. Huhn's proof was very geometric and required a lot of calculations and it is unclear to which extent it can be generalized. But we can show that Huhn's result follows from theorem 2.1 and the following simple observation (cf. [26]): If L is an algebraic lattice and L' its sublattice containing all compact elements, then a lattice identity ϵ holds in L iff it holds in L'. Observing that if L is CJ-generated then (cc_n) is true in L if and only if it is true in L' and that algebraicity was not used to prove 1) $\Rightarrow$ 3) of theorem 2.1, we obtain the following corollary from which the result of Huhn mentioned above follows immediately:

Corollary 2.4 *Let L be an algebraic CJ-generated lattice and L' its sublattice. If L' contains all compact elements, then it is n-distributive iff* (cc_n) *holds.* □

In [25] Nation gave a characterization of n-distributivity which has the same flavor as theorem 2.1. He showed that a variety $\mathcal{V}$ lies in Δ_n if and only if for any $L \in \mathcal{V}$ the following condition σ_n holds: if $x \in J(L)$ and $x \leq \bigvee X, |X| < \infty$, then x is below a join of at most n elements of X. Notice that it is not required that the elements of X be join-irreducible. He also observed that for finite lattices n-distributivity is equivalent to σ_n. It is routine to rework the proof to show that the equivalence holds not only for finite lattices but also for lattices generated by their join-irreducible elements. σ_n can also be used to characterize n-distributivity in arbitrary lattices as follows:

Proposition 2.5 *A lattice L is n-distributive iff its filter lattice $Fil(F)$ satisfies σ_n.*

Proof: If L is n-distributive then so is $Fil(F)$ and σ_n is verified as in the proof of theorem 2.1. Conversely, assume that $Fil(L)$ satisfies σ_n but L is not n-distributive. Then $a = x \wedge \bigvee_{i=0}^{n} y_i >$

$\bigvee_{i=0}^{n}(x \wedge \bigvee_{j\neq i} y_j) = b$ for some $x, y_0, \ldots, y_n$. If f is a maximal filter satisfying $\uparrow a \subseteq f$, $\uparrow b \not\subseteq f$, then f is join-irreducible in $Fil(F)$. The number of filters in the right hand side of $f \leq \uparrow y_0 \vee \ldots \vee \uparrow y_n$ can not be reduced for otherwise we would have $\uparrow b \subseteq f$. This demonstrates a failure of σ_n in $Fil(L)$. $\square$

We will use $\mathbf{D}_n$ and $\mathbf{ID}_n$ to denote the categories of n-distributive and infinitely n-distributive lattices. Infinite n-distributivity requires completeness as the infinite join operation is used in equation $\mathbf{ID}_n$. We define morphisms in $\mathbf{ID}_n$ as lattice homomorphisms preserving infinite joins as well; the morphisms in $\mathbf{D}_n$ are just lattice homomorphisms. It was already stated that any infinitely n-distributive lattice is n-distributive.

Corollary 2.6 *The ideal completion is left adjoint to the forgetful functor from $\mathbf{ID}_n$ to $\mathbf{D}_n$.*

Proof: Given $L \in \mathbf{D}_n$, its ideal completion is n-distributive and algebraic and hence infinitely n-distributive. Given $f : L_1 \to L_2$, define $Idl(f) : Idl(L_1) \to Idl(L_2)$ by making $Idl(f)(\mathcal{I})$ to be the minimal ideal of L_2 that contains $f(\mathcal{I})$. Given $f : L \to L'$ in $\mathbf{D}_n$, define $g = \psi(f)$ by $g(\mathcal{I}) = \bigvee_{x\in\mathcal{I}} f(x)$. Conversely, given $g : Idl(L) \to L'$ in $\mathbf{D}_n$, define $f = \phi(g)$ by $f(x) = g(\downarrow x)$. Clearly, $f = \phi(g)$ is a homomorphism if g is a morphism in $\mathbf{ID}_n$ and $g = \psi(f)$ preserves arbitrary joins if f is a morphism in $\mathbf{D}_n$. We must show $g(\mathcal{I}_1 \cap \mathcal{I}_2) = g(\mathcal{I}_1) \wedge g(\mathcal{I}_2)$. Calculate the right hand side by applying the law of infinite n-distributivity twice:

$$\begin{aligned}
\bigvee_{x\in\mathcal{I}_1} f(x) \wedge \bigvee_{y\in\mathcal{I}_2} f(y) &= \bigvee_{K_2\subseteq\mathcal{I}_2, |K_2|=n} \Big(\bigvee_{x\in\mathcal{I}_1} f(x) \wedge \bigvee_{y\in K_2} f(y)\Big) = \\
&= \bigvee_{K_2\subseteq\mathcal{I}_2, |K_2|=n} \;\bigvee_{K_1\subseteq\mathcal{I}_1, |K_1|=n} \Big(\bigvee_{x\in K_1} f(x) \wedge \bigvee_{y\in K_2} f(y)\Big) = \\
&= \bigvee_{K_1\subseteq\mathcal{I}_1, K_2\subseteq\mathcal{I}_2, |K_2|=|K_1|=n} f\Big(\bigvee K_1 \wedge \bigvee K_2\Big) \leq \bigvee_{x\in\mathcal{I}_1\cap\mathcal{I}_2} f(x)
\end{aligned}$$

which shows $g(\mathcal{I}_1) \wedge g(\mathcal{I}_2) \leq g(\mathcal{I}_1 \cap \mathcal{I}_2)$. The reverse inequality is obvious. This shows that g is a morphism. It is straightforward to check that ϕ and ψ are mutually inverse and establish an adjunction. $\square$

Given a CJ-generated lattice L, define a closure $C_L : \mathbf{2}^{CJ(L)} \to \mathbf{2}^{CJ(L)}$ by $C_L(Y) = CJ(\bigvee Y)$. A closure operator C on a set X is said to be of rank n if $C(Y) = Y$ whenever $C(Y') \subseteq Y$ for any $Y' \subseteq Y$ such that $|Y'| \leq n$ [23]. An easy proof of the following result is left to the reader.

Corollary 2.7 *If L is an algebraic CJ-generated n-distributive lattice, then C_L is of rank n.* $\square$

Algebraic distributive lattices in which the dual infinite distributive law holds are generated by their complete prime elements [24]. This result can be generalized to algebraic n-distributive lattices in which $\mathbf{ID}_n^*$, the dual of $\mathbf{ID}_n$, holds.

Corollary 2.8 *Any n-distributive algebraic lattice L satisfying $\mathbf{ID}_n^*$ is generated by its n-complete primes.*

Proof: L^* is CJ-generated and satisfies $\mathbf{ID}_n$, hence L is coalgebraic. By theorem 2.1 $x = \bigvee(CP_n(L) \cap \downarrow x)$. $\square$

In the rest of the section we turn to the abstract theory of convexity. We augment the standard definition of a convexity by an additional clause saying that intersection of two polytopes is a polytope again (which is true of families of convex sets in vector spaces over ordered division rings) and then define n-dimensional abstract convexities via the Carathéodory condition. Such convexities form a category which is shown to be equivalent to a full subcategory of $\mathbf{D}_{n+1}$. We will return to abstract convexities in section 5.

Definition Given a set X, a *convexity* on X is a family $\mathcal{C}$ of subsets of X (which are called *convex*) such that

- $\emptyset, X \in \mathcal{C}$ (empty set and X are convex);
- $\mathcal{C}$ is closed under arbitrary intersections;
- The union of a directed family of sets of $\mathcal{C}$ is in $\mathcal{C}$;
- $\{x\}$ is in $\mathcal{C}$ for every $x \in X$ (every singleton is convex).

This is the standard definition to which we add one more condition. Given $Y \subseteq X$, its *convex hull* $H_\mathcal{C}$ is defined as the intersection of all $Y' \in \mathcal{C}$ that contain Y.

- If Y_1 and Y_2 are finite subsets of X, then there exist a finite set Y such that $H_\mathcal{C}(Y_1) \cap H_\mathcal{C}(Y_2) = H_\mathcal{C}(Y)$ (intersection of two polytopes is a polytope again).

The usual convexity in $\mathbb{E}^\mathbb{K}$ is the most famous example. For more examples see [10, 18, 30, 32] and Section 5.

We say that a convexity $\mathcal{C}$ *has dimension* n if it satisfies the Carathéodory condition of dimension n (which is actually (cc_{n+1})): If $x \in H_\mathcal{C}(Y)$ where $|Y| > n+1$, then there is an $n+1$-element subset Y' of Y such that $x \in H_\mathcal{C}(Y')$ and n is the minimal number with this property. □

The following belongs to folklore:

Lemma 2.9 *Given a convexity $\mathcal{C}$, its convex sets form a lattice $L(\mathcal{C})$ which is atomistic and algebraic. Moreover, compact elements of $L(\mathcal{C})$ (which are joins of finitely many atoms) form a sublattice of $L(\mathcal{C})$. $L(\mathcal{C})$ is isomorphic to the lattice of closed sets of $H_\mathcal{C}$, closures of finitely many atoms being compact elements.* □

The class of all convexities can be given the structure of a category by defining morphisms as follows: Given two convexities $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$, a morphism $f : (X_1, \mathcal{C}_1) \to (X_2, \mathcal{C}_2)$ is a mapping that maps convex sets to convex sets, preserves arbitrary intersections and directed unions and maps polytopes to polytopes. The category of convexities of dimension n is denoted by $\mathbf{Conv}_n$.

Let $\mathbf{AD}_{n+1}$ be the full subcategory of $\mathbf{D}_{n+1}$ that consists of atomistic lattices in which every element is a finite join of atoms and neither of which satisfies $\mathbf{D}_n$. The following result is reminiscent of the equivalence of the categories of distributive lattices and coherent spaces and coherent maps [15].

Proposition 2.10 *The categories* $\mathbf{Conv}_n$ *and* $\mathbf{AD}_{n+1}$ *are equivalent.*

Proof: Given a convexity $(X,\mathcal{C})$ in $\mathbf{Conv}_n$, let $\Phi((X,\mathcal{C}))$ be the lattice $K(\mathcal{C})$ of compact elements of $L(\mathcal{C})$. Since $L(\mathcal{C})$ is algebraic and atomistic and $(\mathrm{cc})_{n+1}$ holds, $L(\mathcal{C})$ is $n+1$-distributive by theorem 2.1. $K(\mathcal{C})$ is $n+1$-distributive as a sublattice of $L(\mathcal{C})$. It is in $\mathbf{AD}_{n+1}$ because its elements are finite joins of atoms of $L(\mathcal{C})$.

Given a lattice L in $\mathbf{AD}_{n+1}$, define a convexity $(X,\mathcal{C}) = \Psi(L)$ as follows. X is the set of atoms of L and $Y \subseteq X$ is convex if and only if any atom of L which is below $\bigvee Y'$ is in Y whenever Y' is a finite subset of Y.

Both Φ and Ψ can be easily defined for morphisms. Given $f : (X_1,\mathcal{C}_1) \to (X_2,\mathcal{C}_2)$, define $g = \Phi(f) : \Phi((X_1,\mathcal{C}_1)) \to \Phi((X_2,\mathcal{C}_2))$ in $\mathbf{AD}_{n+1}$ as follows. Let $x \in \Phi((X_1,\mathcal{C}_1))$, i.e. x is a compact element of $L(\mathcal{C})$. Then x is a join of atoms, say, $x = a_1 \vee \ldots \vee a_n$, where $a_1,\ldots,a_n$ correspond to elements $x_1,\ldots,x_n \in X_1$. Let X_2' be $f(x_1) \cup \ldots \cup f(x_n)$. Then $g(x)$ is the join of all atoms of $\Phi((X_2,\mathcal{C}_2))$ corresponding to elements of X_2'. Conversely, given a morphism $g : L_1 \to L_2$ in $\mathbf{AD}_{n+1}$, define $f = \Psi(g) : \Psi(L_1) \to \Psi(L_2)$ by $f(Y) = H_{\Psi(L_2)}(\bigcup_{y\in Y} g(\{y\})$ where Y is a subset of the set of atoms of $\Psi(L_1)$.

It is routine to verify that Φ and Ψ are functors which establish an equivalence between the two categories. $\square$

3 Varieties Δ_n and $\Delta_n^{\mathcal{F}}$

In this section we use theorem 2.1 to prove a result which shows that a large class of n-distributive lattices lies in the variety $\Delta_n^{\mathcal{F}}$ generated by the *finite* n-distributive lattices. In fact, all lattices for which the equivalence between n-distributivity and the Carathéodory condition was proved in theorem 2.1 are such. Consequently, we show that two results of this kind proved in [13] are easy corollaries of our theorem.

Theorem 3.1 *Let L be an n-distributive CJ-generated algebraic lattice. Then L is in $\Delta_n^{\mathcal{F}}$.*

Proof: The proof is based on the idea of [13]. Let M be a finite subset of $CJ(L)$. Let L_M be the set of all finite joins of elements of M (including the bottom element 0 of L). Then $\langle L_M, \leq\rangle$ is a finite lattice but not necessarily a sublattice of L. We denote the join and the meet operations of L_M by $\vee^M$ and $\wedge^M$ respectively. Clearly, $x \vee^M y = x \vee y$ and $\bigvee M' \wedge^M \bigvee M'' = \bigvee\{x \in M \mid \exists m' \in M', m'' \in M'' : x \leq m', x \leq m''\}$, $\bigvee\emptyset$ being 0, for any $M', M'' \subseteq M$. Given $x \in L$, define x_M as $\bigvee\{y \mid y \leq x, y \in L_M\}$.

Let $t = t(x_1,\ldots,x_n)$ be a term. By $t^M(x_1^M,\ldots,x_n^M)$ we mean the term that is obtained from t by substituting x_i^M for x_i and changing $\vee$ to $\vee^M$ and $\wedge$ to $\wedge^M$. Let $\mathcal{M}$ be the family of all finite subsets of $CJ(L)$. Our goal is to prove

$$(1) \qquad t(x_1,\ldots,x_n) = \bigvee_{M\in\mathcal{M}} t^M(x_1^M,\ldots,x_n^M)$$

We prove (1) by induction on the number of operations in t. If t is just a variable, $x = \bigvee_{M\in\mathcal{M}} x^M$ because L is CJ-generated. Notice that $x^M \leq x^{M'}$ if $M \subseteq M'$; hence $t^M(x_1^M,\ldots,x_n^M) \leq t^{M'}(x_1^{M'},\ldots,x_n^{M'})$. From this (1) easily follows for $t = t_1 \vee t_2$.

Let $t(x_1,\ldots,x_n)=t_1(x_1,\ldots,x_n)\wedge t_1(x_1,\ldots,x_n)$. We must show the equality

$$(2)\qquad \bigvee_{M\in\mathcal{M}}(t_1^M(x_1^M,\ldots,x_n^M)\wedge^M t_2^M(x_1^M,\ldots,x_n^M))=\bigvee_{M\in\mathcal{M}}t_1^M(x_1^M,\ldots,x_n^M)\wedge\bigvee_{M\in\mathcal{M}}t_2^M(x_1^M,\ldots,x_n^M)$$

since the left hand side of (2) is $\bigvee_{M\in\mathcal{M}}t^M(x_1^M,\ldots,x_n^M)$. First, the $\leq$ inequality clearly holds. To prove the reverse inequality, let z be a completely join-irreducible element which is below the right hand side. Then, for some $M_1,\ldots,M_k\in\mathcal{M}$,

$$z\leq\bigvee_{i=1}^{k}t_l^{M_i}(x_1^{M_i},\ldots,x_n^{M_i}),\ \ l=1,2$$

Let $M=M_1\cup\ldots\cup M_k\cup\{z\}\in\mathcal{M}$. Then $t_l^{M_i}(x_1^{M_i},\ldots,x_n^{M_i})\leq t_l^M(x_1^M,\ldots,x_n^M)$, $l=1,2$. Therefore, $z\leq t_l^M(x_1^M,\ldots,x_n^M)$ for $l=1,2$ and since $z\in M$,

$$z\leq t_1^M(x_1^M,\ldots,x_n^M)\wedge^M t_2^M(x_1^M,\ldots,x_n^M)$$

which finishes the proof of (2). Thus, (1) is proved.

Since L is n-distributive, (cc_n) holds in L. Then, from the definition of L_M it immediately follows that (cc_n) holds in L_M for any $M\in\mathcal{M}$. Thus, each L_M is n-distributive.

Now, let $t_1=t_2$ be an n-ary lattice equation that holds in all finite n-distributive lattices. Then $t_1=t_2$ holds in all lattices L_M. Therefore, $t_1(x_1,\ldots,x_n)=\bigvee_{M\in\mathcal{M}}t_1^M(x_1^M,\ldots,x_n^M)=\bigvee_{M\in\mathcal{M}}t_2^M(x_1^M,\ldots,x_n^M)=t_2(x_1,\ldots,x_n)$ which proves that $L\in\Delta_n^{\mathcal{F}}$. □

From theorem 3.1 we immediately conclude

Corollary 3.2 [13][2] $Co(\mathbb{E}^{n+1})\in\Delta^{\mathcal{F}}_{n}$. □

Notice that only once in the proof of theorem 3.1 did we refer to n-distributivity. It was needed to show that all lattices L_M are n-distributive which in turn was possible because the characterization of n-distributivity restricted to finite lattices does not make use of the $\wedge$ operation. Therefore, theorem 3.1 admits the following generalization. Let $\mathcal{P}$ be a universally quantified first-order sentence in the language that contains $\leq$, $\vee$ and a unary predicate $J(\cdot)$. We say that $\mathcal{P}$ holds in L if $\mathcal{P}$ is true when $\leq,\vee$ and J have obvious interpretations. Let $\mathcal{P}_c$ be obtained from $\mathcal{P}$ by replacing $J(\cdot)$ by $CJ(\cdot)$, the meaning of $CJ(\cdot)$ being "completely join-irreducible". Combining theorem 3.1 with corollary 2.4, we obtain

Corollary 3.3 *Let $\mathcal{P}$ be a universally quantified first-order sentence in the language that contains $\leq,\vee$ and $J(\cdot)$ but does not contain $\wedge$. Assume that a variety $\mathcal{V}$ satisfies the following property: for any CJ-generated algebraic lattice L, $L\in\mathcal{V}$ iff $\mathcal{P}_c$ holds in L. If L is a CJ-generated algebraic lattice and L' its sublattice containing all compact elements, then the following are equivalent:*
1) $L'\in\mathcal{V}$;
*2) $L'\in\mathcal{V}_{\mathrm{fin}}=$ **HSP**(finite members of $\mathcal{V}$);*
3) $L'\models\mathcal{P}_c$. □

[2]To prove this fact in [13], Huhn used another idea which exploited the fact that the compact elements of $Co(\mathbb{E}^{n+1})$ form a sublattice. The proof given in this paper is more general.

In some cases it is possible to use only $\mathcal{P}$ stating assumptions about a variety. For example, let $\mathcal{V}$ be locally finite. Assume that $\mathcal{V}$ satisfies the following property in the spirit of [25]: a variety $\mathcal{V}'$ lies in $\mathcal{V}$ iff $\mathcal{V}' \models \mathcal{P}$ and all finite models of $\mathcal{P}$ are in $\mathcal{V}$. Then it is easy to show that corollary 3.3 remains true for such varieties.

In the rest of the section we give two more corollaries of theorem 3.1. First,

Corollary 3.4 *Let $\mathcal{V}$ be a lattice variety with the following property: Every lattice $L \in \mathcal{V}$ can be embedded into a CJ-generated algebraic lattice $L' \in \mathcal{V}$ and L' can be chosen to satisfy all identities of L. Then $\mathcal{V} \cap \Delta_n = \mathcal{V} \cap \Delta_n^{\mathcal{F}}$.* □

From this corollary the result of [13] stating that $\mathsf{M} \cap \Delta_n = \mathsf{M} \cap \Delta_n^{\mathcal{F}}$ follows immediately since M satisfies the condition of corollary 3.4, see [11].

Since the ideal completion preserves identities, we obtain

Corollary 3.5 *Let L be an n-distributive lattice in which every element is a join of finitely many join-irreducible elements. Then $L \in \Delta_n^{\mathcal{F}}$.* □

4 Examples

In this section we use theorem 2.1 to study n-distributivity in several classes of lattices. The most convenient way to characterize n-distributivity for a lattice L is to calculate its Huhn rank, from now on denoted by $\mathrm{Hn}(L)$. Then L is n-distributive iff $n \geq \mathrm{Hn}(L)$. We consider the following classes of lattices: lattices of finite length, geometric lattices and partition lattices in particular, subsemilattice-lattices, planar lattices and convexity lattices of posets. Convexity lattices are studied separately in Section 5.

Lattices of finite length. Let L be a lattice of finite length and $\ell(L)$ denote its length. Since every element is a join of at most $\ell(L)$ join-irreducible elements, $\mathrm{Hn}(L) \leq \ell(L)$. For finite lattices the result can be made even more precise: the Huhn rank of a finite lattice is at most the width of the poset of its join-irreducible elements.

Geometric lattices. Geometric lattices arise as lattices of closed sets of *matroids* [1]. A *matroid* $\mathbf{M}$ is a pair $\langle S, \overline{(\cdot)}\rangle$, where $\overline{(\cdot)}$ is a closure on S satisfying the exchange axiom ($p \notin \overline{A}, p \in \overline{A \cup \{q\}} \Longrightarrow q \in \overline{A \cup \{p\}}$) and the finiteness of basis condition ($\forall A\, \exists B \subseteq A : |B| \leq \infty$ and $\overline{B} = \overline{A}$). The lattice of closed sets of $\mathbf{M}$ is denoted by $L(\mathbf{M})$. A matroid is called *simple* if the empty set and all one-element sets are closed. We restrict our attention only to simple matroids because for any $\mathbf{M}$, there is a simple matroid $\mathbf{M}'$ such that $L(\mathbf{M})$ and $L(\mathbf{M}')$ are isomorphic [1].

A set $A \subseteq S$ is called *independent* if $p \in \overline{A - \{p\}}$ for no $p \in A$. It is called dependent otherwise. A minimal dependent set is called a *circuit*. It is possible to characterize matroids in terms of circuits [1]. In particular, given the family $\Re$ of the circuits of $\mathbf{M}$, the closure operations can be reconstructed as follows:

$$p \in \overline{A} \Leftrightarrow p \in A \text{ or } \exists C \in \Re : p \in C \subseteq A \cup \{p\}$$

Given a matroid $\mathbf{M}$, let $c(\mathbf{M})$ be the size of the maximal circuits of $\mathbf{M}$.

Theorem 4.1 *Given a simple matroid* $\mathbf{M} = \langle S, \overline{(\cdot)}\rangle$, $\mathrm{Hn}(L(\mathbf{M})) = c(\mathbf{M}) - 1$.

Proof: If C is a circuit of a matroid and $a \in C$, then $C - \{a\}$ is independent. Since sizes of independent sets are bounded above [1], so are the sizes of circuits, i.e. $c = c(\mathbf{M})$ is finite. Since $\mathbf{M}$ is simple, atoms of $L(\mathbf{M})$ correspond to elements of S and we will use the same letter for an element of S and the corresponding atom. Let $a \in S$ and $A \subseteq S$, $a \notin A$. Let $a \leq \bigvee A$. Then $a \in \overline{A}$ and there exists a circuit C such that $a \in C \subseteq A \cup \{a\}$. Therefore, $a \in \overline{C - \{a\}}$. Since $|C| \leq c(\mathbf{M})$, $|C - \{a\}| \leq c(\mathbf{M}) - 1$ which proves the Carathéodory condition with parameter $c(\mathbf{M}) - 1$ for $L(\mathbf{M})$. Since $L(\mathbf{M})$ is algebraic and CJ-generated (in fact, atomistic), by theorem 2.1 it is $c(\mathbf{M}) - 1$-distributive.

Now assume that the Carathéodory condition with parameter $c(\mathbf{M}) - 2$ holds. Let C be a circuit that contains exactly $c(\mathbf{M})$ elements. From the definition of the closure operation it follows that $a \leq \bigvee(C - a)$ for any $a \in C$. By (cc_{c-2}) we find an element $b \in C$, $b \neq a$ such that $a \leq \bigvee(C - \{a, b\})$, that is, $a \in \overline{C - \{a, b\}}$, which contradicts independence of $C - \{b\}$. Therefore, $L(\mathbf{M})$ does not obey $(\mathrm{cc})_{c-2}$ which finishes the proof of $\mathrm{Hn}(L(\mathbf{M})) = c(\mathbf{M}) - 1$. □

Now we can easily prove two consequences of this result. Since a projective geometry can be viewed as a simple matroid underlying matroid induced by the linear closure in a vector space, matroid independence being linear independence, theorem 4.1 tells us that the Huhn rank of a projective geometry is its dimension plus one, cf. [13].

Since $\mathrm{Part}(n)$, the lattice of partitions of an n-element set, is the lattice of closed sets of the polygon matroid of a complete graph with n vertices, we conclude

Corollary 4.2 $\mathrm{Hn}(\mathrm{Part}(n)) = n - 1$. □

More generally, for any finite graph the Huhn rank of the lattice of closed sets of its polygon matroid is one less than the size of the maximal circuit.

A result similar to theorem 4.1 can be proved for antimatroids, or convex geometries [10]. Recall that a convex geometry on a finite set S is a closure $\overline{(\cdot)}$ satisfying the *antiexchange* property: $p, q \notin \overline{A}, p \in \overline{A \cup \{q\}} \Longrightarrow q \notin \overline{A \cup \{p\}}$. A subset A of S is called free if every its subset is the intersection of A with a complement of a closed set. Minimal nonfree sets are called *circuits* of the convex geometry. It is possible to characterize convex geometries in terms of circuits [8] and the Carathéodory type result was proved for the anti-exchange closures in [18]. It is rather straightforward to extend the result of [18] to lattices of closed sets of convex geometries. Thus, we have:

Corollary 4.3 *Let* $\langle S, \overline{(\cdot)}\rangle$ *be a convex geometry and* L *the lattice of closed sets. Let* c *be the maximal size of a circuit. Then* $\mathrm{Hn}(L) = c - 1$. □

Lattices of subsemilattices. Let $\langle S, \sqcup\rangle$ be a join semilattice. By $\mathrm{Sub}(S)$ we denote the lattice of subsemilattices of S. Let $F(n)$ denote the free semilattice with n generators, that is, the semilattice of all nonempty subsets of an n-element set ordered by inclusion.

Proposition 4.4 *Given a semilattice* S*, the lattice of its subsemilattices* $\mathrm{Sub}(S)$ *is* n*-distributive iff* S *does not contain a subsemilattice isomorphic to* $F(n+1)$.

Proof: Suppose $\mathrm{Sub}(S)$ is not n-distributive. Then (cc_n) does not hold and there exists $k > n$ such that $\{a\} \leq \{a_1\} \vee \ldots \vee \{a_k\}$ but for no i is $\{a\}$ below $\bigvee_{j \neq i}\{a_j\}$. Here a and a_i's are elements of S. This implies $a = a_1 \sqcup \ldots \sqcup a_k$ but $a \neq \bigsqcup_{i \in I} a_i$ for any proper subset I of $\{1, \ldots, k\}$. Assume that for two different subsets I_1, I_2 of $\{1, \ldots, k\}$ it holds: $\bigsqcup_{i \in I_1} a_i = \bigsqcup_{i \in I_2} a_i$. Without loss of generality, let $i \in I_1 - I_2$. Then $a = a_1 \sqcup \ldots \sqcup a_k = \bigsqcup_{j \neq i} a_j$, a contradiction. Hence, the subsemilattice generated by $a_1, \ldots, a_k$ is isomorphic to $F(k)$ and then $F(n+1)$ is a subsemilattice of S. Conversely, if S' is a subsemilattice of S isomorphic to $F(n+1)$, let $a_1, \ldots, a_{n+1}$ be its atoms and a its top. Then $\{a\} \leq \{a_1\} \vee \ldots \vee \{a_{n+1}\}$ but $\{a\} \not\leq \bigvee_{j \neq i}\{a_j\}$ for any i. Hence, (cc_n) does not hold and $\mathrm{Sub}(S)$ is not n-distributive. $\square$

As the first corollary we obtain the result of [21] that $\mathrm{Sub}(S)$ is distributive if and only if S is a chain. Another corollary of proposition 4.4 deals with dimension. The n-dimensional Euclidean space can be considered as a semilattice with the ordering being componentwise, the join operation being max. Since $\langle \mathbb{E}^{\Bbbk}, \max\rangle$ contains a subsemilattice isomorphic to $F(n)$ but no subsemilattice isomorphic to $F(n+1)$, we obtain

Corollary 4.5 $\mathrm{Sub}(\langle \mathbb{E}^{\Bbbk}, \max\rangle) \in \gneqq_{\Bbbk} - \gneqq_{\Bbbk - \nVdash}$. $\square$

Finding a characterization of n-distributivity in the lattices of sublattices for an arbitrary n remains open. For 2-distributivity see [7].

Planar lattices. Planarity is closely related to the order dimension. Given a poset $\langle P, \sqsubseteq\rangle$, its dimension, $\dim(\langle P, \sqsubseteq\rangle)$, is the minimal number of chains whose product contains $\langle P, \sqsubseteq\rangle$ as a subposet, see [17]. First we demonstrate how dim and Hn are related.

Proposition 4.6 *Let L be a finite lattice. Then* $\mathrm{Hn}(L) \leq \dim(L)$.

Proof: Suppose that there exists a finite lattice L such that $\mathrm{Hn}(L) > \dim(L) = n$. Then (cc_n) does not hold, i.e. there exists a number $k > n$ and $k+1$ join-irreducible elements $a, a_1, \ldots, a_k$ such that $a \leq a_1 \vee \ldots \vee a_k$ but $a \not\leq \bigvee_{j \neq i} a_j$ for all $i = 1, \ldots, k$. Clearly, neither of a_i's is the bottom element of L and $\bigvee_{i \in I} a_i \neq \bigvee_{j \in J} a_j$ whenever I and J are distinct subsets of $\{1, \ldots, k\}$ (cf. the proof of proposition 4.4). Consider the subposet of L formed by the bottom element and all joins $\bigvee_{i \in I} a_i$ where $\emptyset \neq I \subseteq \{1, \ldots, k\}$. From the above observation it follows that this subposet if isomorphic to $\mathbf{1} \oplus F(k)$, i.e. $\mathbf{2}^k$, the lattice of subsets of a k-element set. This lattice is known to have dimension k [17], hence $\dim(L) \geq k > n$, a contradiction. This contradiction shows $\mathrm{Hn}(L) \leq \dim(L)$. $\square$

Since finite lattice is planar if and only if its dimension is ≤ 2 [16], we conclude

Corollary 4.7 *Any finite planar lattice is either distributive or 2-distributive.* $\square$

Convexity lattices of posets. Given a poset $\langle P, \sqsubseteq\rangle$, its subset is called convex if it includes, together with $x \sqsubseteq y$, any element z such that $x \sqsubseteq z \sqsubseteq y$. The lattice of convex subsets of P is called its convexity lattice and denoted by $Co(P)$, see [5]. It was proved in [5] that $Co(P)$ is atomistic, algebraic and its Carathéodory rank is at most 2. Therefore, $Co(P)$ is either 1- or 2-distributive. To characterize its Huhn rank it is enough to describe those posets P for which $Co(P)$ is distributive. Let P contain a nonsimple interval $[x, y]$ and $z \in [x, y]$, $z \neq x, y$. Then $\{z\} \sqsubseteq \{x\} \sqcup \{y\}$ in $Co(P)$ which shows that (cc_1) fails. Obviously, (cc_1) holds if all intervals are simple. Thus, we have

Proposition 4.8 *Given a poset $\langle P, \sqsubseteq \rangle$, its convexity lattice $Co(P)$ is distributive or 2-distributive. It is distributive iff P is of length 0 or 1.* $\square$

5 Convexity lattices

In this section we study n-distributivity and dual n-distributivity in convexity lattices. We will show that the Huhn rank of a convexity lattice coincides with its *affine rank* defined as the height of the lattice of "affine flats" (in fact, the height of the modular core). Under natural assumptions about the properties of the underlying betweenness relation convexity lattices of dimension n (equivalently, of affine rank $n+1$) arise as lattices of convex sets of convexities of dimension n (see section 2 for the definition). Finally, we will relate the *dual* n-distributivity to dimension in convexity lattices. We start with some terminology.

Definition [3] An atomistic lattice is called *biatomic* if $p \leq x \vee y$ where p is an atom and x, y are nonzero implies $p \leq x' \vee y'$ where $x' \leq x$ and $y' \leq y$ are atoms.

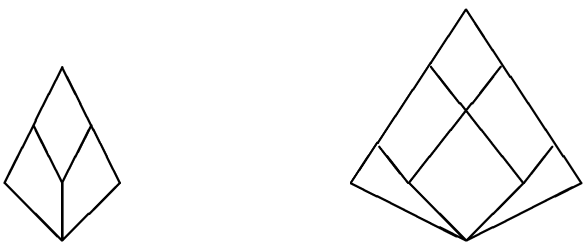
Given a lattice L, $\langle a_1, \dots, a_n \rangle$ denotes the sublattice of L generated by $a_1, \dots, a_n \in L$.

Definition [4] A biatomic algebraic lattice L is called a convexity lattice if it satisfies the following properties CL1 and CL2:

CL1 If p, q, r are distinct atoms, then $\langle p, q, r \rangle$ is isomorphic to $\mathbf{2}^3$ or $Co(\underline{3})$;

CL2 If p, q, r, s are distinct atoms and both $\langle p, q, r \rangle$ and $\langle q, r, s \rangle$ are isomorphic to $Co(\underline{3})$, then $\langle p, q, r, s \rangle$ is isomorphic to $Co(\underline{4})$.

$Co(\underline{n})$ is the lattice of intervals of an n-element chain. The diagrams of $Co(\underline{3})$ and $Co(\underline{4})$ are shown below:



$Co(\underline{3})$ $\qquad$ $Co(\underline{4})$

The conditions CL1 and CL2 can be better understood if one thinks in terms of the betweenness relation β. If three points are non-collinear, i.e. they form a triangle, the lattice of convex sets of such a configuration is $\mathbf{2}^3$. If they are collinear, i.e. one of them is between the others, the lattice of convex sets is $Co(\underline{3})$. The condition CL2 says that if two triples of points, (p, q, r) and (q, r, s) are collinear, then all four are collinear. The conditions CL1 and CL2 imply, in particular, that the closure on the set of atoms induced by L satisfies the antiexchange property.

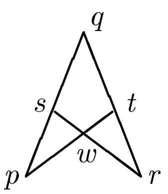
Usually the definition of convexity lattices is augmented by properties reminiscent of Hilbert's order axioms for the betweenness[3]. To introduce them, some preliminary work needs to be done.

An element a of a lattice L is called *modular* if, for any $x \in L$, $c \leq a$ implies $c \vee (x \wedge a) = (c \vee x) \wedge a$. The set of modular elements is denoted by $M(L)$. The following results appeared in [4]: If L is the lattice $Co(V)$ of convex sets in a vector space V over an ordered division ring, $M(Co(V))$ is the meet-subsemilattice of affine flats. If L is a convexity lattice, $M(L)$ is closed under arbitrary meets and $1 \in M(L)$. Define

$$x \nabla y \stackrel{\mathsf{def}}{=} \bigwedge (M(L) \cap \uparrow (x \vee y))$$

Then $\langle M(L), \nabla, \wedge \rangle$ is an algebraic atomistic lattice, its atoms being the atoms of L. If p, q are distinct atoms, $p \nabla q$ is called a *line*. A line given by p and q consists of all atoms r such that $\langle p, q, r \rangle \cong Co(\underline{3})$. In other words, $p \nabla q$ consists of all atoms r such that $r \leq p \vee q$ or $p \leq q \vee r$ or $q \leq p \vee r$.

A convexity lattice is said to be a *Peano convexity lattice* if for distinct atoms p, q, r, s, t such that $s \leq p \vee q$ and $t \leq q \vee r$ there exists an atom $w \leq (s \vee r) \wedge (p \vee t)$, see the picture below.



A convexity lattice is said to have the *divisibility property* if for any two distinct atoms p and q there exists an atom $r \leq p \vee q$, $r \neq p, q$. It is called *unbounded* [20, 22] if for any p and q there exists an atom r such that $p \leq r \vee q$ (this is reminiscent of Hilbert's axiom II_2). Equivalently, a convexity lattice is unbounded if 0 and 1 are the only codistributive elements [20, 22].

Any convexity lattice with the divisibility property is Peano. If L is a Peano convexity lattice, $M(L)$ has the exchange property [4]. Hence, if it is of finite length, it is a geometric lattice and its length is denoted by $\mathrm{aff}(L)$ and is called the *affine rank* of L. If L is unbounded and $\mathrm{aff}(L) > 2$, then L has the divisibility property [22].

Given a convexity lattice L with the set of atoms A, define $\mathcal{C}_L \subseteq \mathbf{2}^A$ to be the family of sets of atoms under elements of L, i.e. $X \in \mathcal{C}_L$ if and only if there exists $x \in L$ such that X is the set of atoms below x. Now we are ready to prove the first result of this section.

Theorem 5.1 *Let L be a convexity lattice of affine rank n satisfying the divisibility property and A the set of its atoms. Then $(A, \mathcal{C}_L)$ is an $n-1$-dimensional convexity.*

Proof: We need a few auxiliary definitions first. By $A(x)$ we mean the set of atoms below x. If L is a convexity of finite affine rank n, a coatom of $M(L)$ is called a *hyperplane*. Given a hyperplane h and

[3]Axiomatization of elementary geometry in terms of the betweenness relation was given by Tarski [31]. One can consult that work or [4] for the motivation for the conditions to be introduced.

two atoms p and q, either $p \vee q \leq h$ or $(p \vee q) \wedge h = 0$ or $(p \vee q) \wedge h$ is an atom (if $(p \vee q) \wedge h$ contained two atoms, it would contain the whole line $p \nabla q$).

Define a relation E_h on A by $pE_hq \Leftrightarrow p \vee q \leq h$ or $(p \vee q) \wedge h = 0$. Then E_h is an equivalence relation having two or three equivalence classes, $A(h)$ being one of them [2, 22]. We denote the equivalence classes different from $A(h)$ by h^+ and h^-. h^- may not exist. $\mathbf{h}^+ \stackrel{\text{def}}{=} h \vee h^+$ and $\mathbf{h}^- \stackrel{\text{def}}{=} h \vee h^-$ are called the *closed halfspaces* [22]. $A(\mathbf{h}^*) = A(h) \cup A(h^*)$, where $* \in \{+, -\}$. Given atoms $p_1, \ldots, p_n$, $d = p_1 \vee \ldots \vee p_n$ is called a *simplex* [22] if $p_1 \nabla \ldots \nabla p_n = 1$. Its ith side is $d^i = \bigvee_{j \neq i} p_j$.

If x is an element of L, ∇x is the minimal element of $M(L)$ above x, i.e. $\bigwedge(y \in M(L) \mid y \geq x)$.

It is clear that the lattice of convex sets of $(A, \mathcal{C}_L)$ is L. To prove that $(A, \mathcal{C}_L)$ is $n-1$-dimensional, (cc_n) must be shown to hold in L. But this follows from [4, theorem 19]. Thus, it is enough to show that the compact elements of L form a sublattice. We start with two claims.

Claim 1: Let d be a simplex and h a hyperplane. Then $d \wedge \mathbf{h}^+$ is compact.
Proof of claim 1: If h^- does not exist, $d \wedge \mathbf{h}^+ = d$ which is a compact element. Assume that h^- exists. Let $d = p_1 \vee \ldots \vee p_n$ where $p_1 \nabla \ldots \nabla p_n = 1$. If all p_i's are under h^-, $0 = d \wedge \mathbf{h}^+$ is a compact element. Now, let $p_1, \ldots, p_k \in A(\mathbf{h}^+)$ and $p_{k+1}, \ldots, p_n \in A(h^-)$. For any $i \leq k$ and $j > k$ define $p_{ij} = (p_i \vee p_j) \wedge h$. According to the definition of E_h, $p_{ij} \neq 0$. Moreover, it follows from the properties of the modular core elements that p_{ij} is an atom, cf. [22]. Let

$$d' = p_1 \vee \ldots \vee p_k \vee \bigvee_{i \leq k, j > k} p_{ij}.$$

We claim $d' = d \wedge \mathbf{h}^+$. Clearly, $d' \leq d \wedge \mathbf{h}^+$. To prove the reverse inequality, let $v \leq d \wedge \mathbf{h}^+$, $v \in A$. Then, by biatomicity, there exist atoms $q \leq p_1 \vee \ldots \vee p_k$ and $r \leq p_{k+1} \vee \ldots \vee p_n$ such that $v \leq q \vee r$. Since $q \leq \mathbf{h}^+$ and $r \leq h^-$, $w = (q \vee r) \wedge h$ is an atom and $v \leq w \vee q$. If w does not coincide with one of p_{ij}'s, consider the line $w \nabla p_{ij}$. By [4, theorem 10] there exists an index i_0 and an atom $s \leq d^{i_0}$ such that $w \leq p_{ij} \vee s$. Since $s \leq p_{ij} \nabla w \leq h$, this shows $w \leq \bigvee_{l=1}^{n} (d^l \wedge h)$. Now, $d^l \wedge h = (h \wedge (\nabla d^l)) \wedge d^l$. If $\nabla d^l \leq h$, then $d^l \leq p_1 \vee \ldots \vee p_k$. If $h \not\geq \nabla d^l$, then $h \wedge \nabla d^l$ is a hyperplane in ∇d^l because $M(L)$ is a geometric lattice, and $h \wedge d^l = \bigvee_m (h \wedge d^{lm})$. Continuing this process, we finally obtain $w \leq p_1 \vee \ldots \vee p_k \vee \bigvee_{(l,m)} (h \wedge (p_l \vee p_m))$ where (l,m)'s range over a set of pairs of indices. Since $h \wedge (p_l \vee p_m)$ is either $p_l \vee p_m$ (and then $l, m \leq k$) or 0 or p_{lm}, this shows $w \leq d'$ and $v \leq d'$ as $q \leq d'$. Hence, $d' = d \wedge \mathbf{h}^+$. Since d' is the join of finitely many atoms, it is compact. Claim 1 is proved.

Using claim 1, we can prove the following
Claim 2: If x is a compact element and h is a hyperplane, then $x \wedge \mathbf{h}^+$ is compact.
Proof of claim 2: Assume without loss of generality that $\nabla x = 1$ (if this is not the case, consider $h' = h \wedge (\nabla x)$. Then, if $x \not\leq h$, h' is a hyperplane in $\downarrow \nabla x)$. Since $\mathrm{aff}(L) = n$, the Carathéodory condition (cc_n) holds [4]. Therefore, there exist simplexes $d_1, \ldots, d_l$ such that $x = d_1 \vee \ldots \vee d_l$ and, moreover, $A(x) = A(d_1) \cup \ldots \cup A(d_l)$. Let $x_i = d_i \wedge \mathbf{h}^+$. Then $x \wedge \mathbf{h}^+ = \bigvee_i x_i$ which proves compactness of $x \wedge \mathbf{h}^+$.

Now, let x, y be two compact elements. Since $x \wedge y = (x \wedge (\nabla x)) \wedge (y \wedge (\nabla x))$, we may assume without loss of generality that $\nabla x = 1$. Again, $A(x) = \bigcup_{i=1}^{l} A(d_i)$ and $x = \bigvee_{i=1}^{n} d_i$ where d_i's are simplexes. According to [22, theorem 15], for each simplex d_i there exist n hyperplanes h_{ij}, $j = 1, \ldots, n$ such that $d_i = \bigwedge_j \mathbf{h}_{ij}^+$. Then, according to claim 2, $d_i \wedge y = y_i$ is a compact element. We claim $x \wedge y = \bigvee_i y_i$.

Clearly, $y_i \leq x \wedge y$. Conversely, given an atom $p \leq x \wedge y$, there exists an index i such that $p \leq d_i$. Hence, $p \leq y_i$. Thus, $x \wedge y$ is compact, which finishes the proof of the theorem. $\square$

Corollary 5.2 *Given a convexity lattice L with the divisibility property,* $\mathrm{aff}(L) = \mathrm{Hn}(L)$. $\square$

In the rest of the section we will show that the affine rank can be characterized via the *dual* n-distributivity as well. The key lemma establishes the relationship between the dual n-distributivity and the Helly condition of dimension n in a class of lattices that, as we shall show, includes many convexity lattices. The Helly condition of dimension n, reminiscent of Helly's theorem, reads as follows:

Let L be a lattice with 0 and $x_1, \ldots, x_k \in L$, $k > n+1$. Then $\bigwedge_{i=1}^{k} x_i \neq 0$ whenever $\bigwedge_{j=1}^{n+1} x_{i_j} \neq 0$ for any sequence $i_1, \ldots, i_{n+1}$ of indices.

Lemma 5.3 *Let L be a biatomic algebraic lattice satisfying the following property: If x_0, x_1, y_0, y_1 are atoms and p is an atom below $x_i \vee y_i$ for $i = 0, 1$, then for any atom $x \leq x_0 \vee x_1$ there exists an atom $y \leq y_0 \vee y_1$ such that $p \leq x \vee y$. Then L is dually n-distributive if the Helly condition of dimension $n-1$ holds.*

Proof: We first prove that the condition of lemma implies the following, more general property: If $x_0, \ldots, x_k, y_0, \ldots, y_k$ are atoms and p is an atom below $x_i \vee y_i$ for all $i = 1, \ldots, k$, then for any atom $x \leq x_0 \vee \ldots \vee x_k$ there exists an atom $y \leq y_0 \vee \ldots \vee y_k$ such that $p \leq x \vee y$. The proof is by induction on k. For $k = 1$ this is the condition of lemma. For an arbitrary k, by biatomicity there exists an atom $x' \leq x_1 \vee \ldots \vee x_k$ such that $x \leq x_0 \vee x'$. By induction hypothesis, there exists an atom $y' \leq y_1 \vee \ldots \vee y_k$ such that $p \leq x' \vee y'$. Then there exist an atom $y \leq y_0 \vee y' \leq y_0 \vee \ldots \vee y_k$ such that $p \leq x \vee y$.

Let the Helly condition of dimension $n-1$ hold. To prove that L is dually n-distributive, it is enough to show that for any atom p,

$$p \leq \bigwedge_{i=0}^{n} (x \vee \bigwedge_{j \neq i} y_j) \text{ implies } p \leq x \vee \bigwedge_{i=0}^{n} y_i.$$

Let p be below the left hand side. If any $\bigwedge_{j \neq i} y_j$ is 0, then p is trivially under x. Assume $\bigwedge_{j \neq i} y_j \neq 0$ for all i. Then for any i there exist atoms $p_i \leq x$ and $q_i \leq \bigwedge_{j \neq i} y_j$ such that $p \leq p_i \vee q_i$. Define y'_i as $\bigvee_{j \neq i} q_j \leq y_i$. Then $q_i \leq \bigwedge_{j \neq i} y'_j$. By the Helly condition, there exists an atom $q \leq \bigwedge_{i=0}^{n} y'_i$. Then $q \leq q_0 \vee \ldots \vee q_{n-1}$ and there exists an atom $r \leq p_0 \vee \ldots \vee p_{n-1} \leq x$ such that $p \leq r \vee q \leq x \vee \bigwedge_{i=0}^{n} y'_i \leq x \vee \bigwedge_{i=0}^{n} y_i$, proving dual n-distributivity. $\square$

Theorem 5.4 *Let L be an unbounded convexity lattice of affine rank n, $n \geq 3$. Then L is dually n-distributive but not dually $n-1$-distributive.*

Proof: Since $\mathrm{aff}(L) \geq 3$, L has the divisibility property [22]. Therefore, L satisfies the condition of lemma 5.3, see [22, lemma 1]. According to [4], the Helly condition of dimension $n-1$ is true in L. Therefore, L is dually n-distributive by lemma 5.3.

To show that L is not $n-1$-distributive, notice that the Helly condition of dimension $n-2$ does not hold [4]. Therefore, there exist $y_1, \ldots, y_n \in L$ such that each $\bigwedge_{j \neq i} y_j$ contains an atom q_i but

$\bigwedge_{i=1}^{n} y_i = 0$. Some q_i's may be the same. Let $\{q_1, \ldots, q_k\}$ be distinct elements of $\{q_1, \ldots, q_n\}$, $k \leq n$. Clearly, we can assume that $k > 1$ for otherwise q_1 would be in $\bigwedge_i y_i$. Using the antiexchange property, it is easy to show that there exists q_i which is not under the join of all q_j's, $j \neq i$. Without loss of generality, let $i = 1$. Since L is unbounded, find an atom r_i such that $q_1 \leq r_i \vee q_i$, $i = 2, \ldots, k$. Let $x = r_2 \vee \ldots \vee r_k$. If $q_1 \leq x$, then $q_1 \leq r_2 \vee r_2'$ where r_2' is an atom under $r_3 \vee \ldots \vee r_k$. By the property proved in lemma 5.3, there exists an atom $q_2' \leq q_3 \vee \ldots \vee q_k$ such that $q_1 \leq r_2' \vee q_2'$. Then from CL2 it follows that $q_1, q_2, r_2, q_2', r_2'$ lie on the same line and then it is easy to show that $q_1 \leq q_2 \vee q_2' \leq q_2 \vee \ldots \vee q_k$, a contradiction. Hence, $q_1 \not\leq x$.

Let $y_i' = \bigvee_{j \neq i} q_j$. Then $q_i \leq \bigwedge_{j \neq i} y_i'$ and $\bigwedge_i y_i' \leq \bigwedge_i y_i = 0$. We have: $x \vee \bigwedge_{i=1}^{n} y_i' = x \not\geq q_1$ but $q_1 \leq x \vee \bigwedge_{j \neq i} y_j'$ for any $i = 1, \ldots, n$, hence $q_1 \leq \bigwedge_{i=1}^{n} (x \vee \bigwedge_{j \neq i} y_j')$. Therefore, L is not $n-1$-distributive. $\square$

The assumption $\mathrm{aff}(L) \geq 3$ was needed only in order to prove that L has the divisibility property. Since the divisibility property is true in $Co(\mathbb{E}^{\ltimes})$ for an arbitrary n, we obtain

Corollary 5.5 [13] *The dual of $Co(\mathbb{E}^{\ltimes})$ is in $\Delta_{n+1} - \Delta_n$.* $\square$

6 Concluding remarks

In this paper we have developed the idea of [13] that dimension can be expressed as an algebraic property of lattices of convex sets. We have proved that in a large class of lattices (algebraic lattices in which every element is the join of completely join irreducible elements) the lattice theoretic form of Carathéodory's theorem is equivalent to n-distributivity . Moreover, such lattices are n-distributive if and only if they are in the variety generated by the finite n-distributive lattices. These results were applied to characterize n-distributivity in various classes of lattices. For example, in a geometric lattice it is the size of the maximal circuit of the underlying matroid that determines the least n such that the lattice is n-distributive. In convexity lattices, which are a generalization of lattices of convex sets, the dual n-distributivity determines dimension as well.

A few questions remain open. One of them was mentioned already. While a concise characterization of n-distributivity of subsemilattice-lattices is easy to obtain, it is not known whether a similar result can be proved for sublattice-lattices.

The lattices of convex sets (and even convexity lattices with the divisibility property) are n-distributive iff they are dually n-distributive. Since Carathéodory's theorem is equivalent to n-distributivity and Helly's theorem implies the dual n-distributivity, this suggests that there may exist a lattice theoretic duality between Carathéodory's and Helly's theorems. This is not a mere speculation. Indeed, take a convexity lattice L of affine rank n with the divisibility property. Then it is dually n-distributive which means its dual is n-distributive. The dual of any algebraic lattice is CJ-generated. Now, if we notice that algebraicity was not used to prove 1) $\Rightarrow$ 3) of theorem 2.1, we conclude that (cc_n) holds in the dual of L, i.e. Helly's theorem of dimension n implies the dual of Carathéodory's theorem of the same dimension. This kind of duality will be further investigated.

n-distributivity was first introduced and studied for modular lattices. It was observed that it allows us to characterize the dimension of a projective geometry in the way similar to the one exploited in this paper. Another sequence of dimension discriminating equations for projective geometries was given in [9]. In is not clear, however, to what extent the results of this paper can be generalized if equations of [9] are used.

Finally, several algebraic models of convexity have been proposed recently, e.g. [10, 28, 30, 32]. We believe that investigation of the relationship between Carathéodory's and Helly's theorems and (dual) n-distributivity in those models may lead to new intersting results.

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References

[1] M. Aigner, *"Combinatorial Theory"*, Springer Verlag, Berlin, 1979.

[2] M.K. Bennett, Separation condition on convexity lattices, in: *"Universal Algebra and Lattice Theory"* (S. Comer, ed), Springer Lecture Notes in Mathematics, **1149** (1984), 22-36.

[3] M.K. Bennett, Biatomic lattices, *Algebra Universalis* **24** (1987), 60-73.

[4] M.K. Bennett and G. Birkhoff, Convexity lattices, *Algebra Universalis* **20** (1985), 1-26.

[5] M.K. Bennett and G. Birkhoff, The convexity lattice of a poset, *Order* **2** (1985), 223-242.

[6] P. Crawley and R. Dilworth, *"Algebraic Theory of Lattices"*, Prentice-Hall, 1973.

[7] G. Czédli, On the 2-distributivity of sublattice lattices, *Acta Math. Acad. Sci. Hungar.* **36** (1980), 49-55.

[8] B. Dietrich, A circuit set characterization of antimatroids, *J. Combin. Th. (B)* **43** (1987), 314-321.

[9] A. Day, Dimension equations in modular lattices, *Algebra Universalis* **22** (1986), 14-26.

[10] P.H. Edelman, R.E. Jamison, The theory of convex geometries, *Geom. Dedicata* **19** (1985), 247-270.

[11] U. Faigle, Frink's theorem for modular lattices, *Arch. Math.* **36** (1981), 179-182.

[12] A. Huhn, Schwach distributive Verbände, *Acta Sci. Math. (Szeged)* **33** (1972), 297-305.

[13] A. Huhn, On non-modular n-distributive lattices: Lattices of convex sets, *Acta Sci. Math. (Szeged)* **52** (1987), 35-45.

[14] G. Grätzer, *"General Lattice Theory"*, Academic Press, New York, 1978.

[15] P.T. Johnstone, *"Stone Spaces"*, Cambridge University Press, 1982.

[16] D. Kelly, I. Rival, Planar lattices, *Canadian J. Math.* **27** (1975), 635-665.

[17] D. Kelly, W.T. Trotter, Dimension theory for ordered sets, in: *"Ordered Sets"* (I. Rival, ed.), D. Reidel Publishing Company, 1981, pages 171-212.

[18] B. Korte, L. Lovász and R. Schrader, *"Greedoids"*, Springer-Verlag, Berlin, 1991.

[19] L. Libkin, On the characterization of non-modular n-distributive lattices, *Preprint No. 18, Mathematical Institute*, Budapest, 1989.

[20] L. Libkin, Parallel axiom in convexity lattices, *Periodica Mathematica Hungarica* **24** (1992), 1-12.

[21] L. Libkin, I. Muchnik, On a subsemilattice-lattice of a semilattice, *MTA SZTAKI Közlemények* **39** (1988), 101-110.

[22] L. Libkin, I. Muchnik, Halfspaces and hyperplanes in convexity lattices, *Preprint No. 51, Mathematical Institute*, Budapest, 1989.

[23] R. McKenzie, G. McNulty, W. Taylor, *"Algebras, Lattices, Varieties"*, volume I, Wadsworth and Brookes/Cole, Monterey, California, 1987.

[24] M. Mislove, When are order scattered and topologically scattered the same? *Ann. Discrete Math.* **23** (1984), 61-80.

[25] J.B. Nation, An approach to lattice varieties of finite height, *Algebra Universalis* **27** (1990), 521-543.

[26] P.P. Palfy, Modular subalgebra lattices, *Algebra Universalis* **27** (1990), 220-229.

[27] R.T. Rockafellar, *"Convex Analysis"*, Princeton University Press, 1970.

[28] A. Romanowska, J.D.H. Smith, *"Modal Theory: An Algebraic Approach to Order, Geometry and Convexity"*, Heldermann Verlag, Berlin, 1985.

[29] G. Richter, On the structure of lattices in which every element is a join of join-irreducible elements, *Periodica Mathematica Hungarica* **13** (1982), 47-69.

[30] V.P. Soltan, *"Introduction to the Axiomatic Theory of Convexity"*, Kishinev, Ŝtiinca, 1984.

[31] A. Tarski, What is elementary geometry?, in: *"The Axiomatic Method, with Special Reference to Geometry and Physics*, (L. Henkin, ed.) North Holland, Amsterdam, 1959, pages 16-29.

[32] M. Van de Vel, Binary convexities and distributive lattices, *Proc. London Math. Soc.* **48** (1984), 1-33.