This assignment is due at the **start** of the lecture on Friday, 24 January 2020.

The purpose of the assignment is to check that you are following what we've been talking about in class this week.

1. [5 marks]

   Show that, for any vector norm $\| \cdot \|$ for $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

   $$| \, \|x\| - \|y\| \, | \leq \|x - y\|$$

2. [5 marks]

   Sometimes you see the *matrix norm subordinate to a vector norm* defined as

   $$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \tag{1}$$

   and other times you see it defined as

   $$\|A\| = \max_{\|x\|=1} \|Ax\| \tag{2}$$

   Show that

   $$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\| \tag{3}$$

   So definitions (1) and (2) are equivalent.

   Note that A may be an $m \times n$ matrix, with $m \neq n$. In this case, the norm for x in equations (1), (2) and (3) is a vector norm for $\mathbb{R}^n$, while the norm for Ax in equations (1), (2) and (3) is a vector norm for $\mathbb{R}^m$. Hence, these two norms are different. Therefore, in your proof, don't assume that the vector norm for x is the same as the vector norm for Ax. However, the vector norm for x is the same everywhere it occurs in equations (1), (2) and (3). Similarly, the vector norm for Ax is the same everywhere it occurs in equations (1), (2) and (3).

   Your proof should hold for any vector norm for $\mathbb{R}^m$ and any vector norm for $\mathbb{R}^n$.

3. [5 marks]

   Show that, for any matrix $A \in \mathbb{R}^{m \times n}$ and any vector $x \in \mathbb{R}^n$,

   $$\|Ax\| \leq \|A\| \, \|x\|$$

   where $\|A\|$ is the matrix norm for A defined by (1) or equivalently (2) and $\|Ax\|$ and $\|x\|$ are the vector norms for Ax and x used in (1) and (2).

4. [5 marks]

   Show that there is always an $x^* \neq 0$ such that

$$\|Ax^*\| = \|A\| \, \|x^*\| \tag{4}$$

   where again $\|A\|$ is the matrix norm for A defined by (1) or equivalently (2) and $\|Ax\|$ and $\|x\|$ are the vector norms for Ax and x used in (1) and (2).

   Moreover, show that, if x^* satisfies (4), then $\hat{x} = \alpha x^*$ satisfies

$$\|A\hat{x}\| = \|A\| \, \|\hat{x}\|$$

   for all $\alpha \in \mathbb{R}$.

   Hence, there are infinitely many x^* that satisfy (4).

5. [5 marks]

   Show that an *isolated local minimizer* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a *strict local minimizer* of f.

   (See pages 13 and 14 of your textbook for the definitions of *isolated local minimizer* and *strict local minimizer* of f.)