Conclusion of N&W proof on page 91 of the textbook.

On page 91 of their textbook, N&W derive the inequality

$$\frac{1}{2}(p-p^*)^T (B+\lambda I)(p-p^*) \geq 0 \qquad (4.51)$$

and then they say that the set of directions

$$S = \left\{ w : \quad w = \pm \frac{p-p^*}{\|p-p^*\|_2} \text{ for some } p \text{ with } \|p\|_2 = \Delta \right\}$$

is dense on the unit sphere. So (4.51) suffices to show that $B+\lambda I$ is positive semi-definite.

Yesterday, in our lecture, I wasn't quite sure why (4.51) implies (4.8c) (i.e. $B+\lambda I$ pos. semi-def.)

The problem is, the set S does not contain all u s.t. $\|u\|=1$. To see this, suppose $u^T p^* = 0$

Now suppose there is a $p \in R^n$ s.t. $\|p\|_2 = \Delta$ and

$$u = \frac{p-p^*}{\|p-p^*\|_2} \implies p = p^* + \|p-p^*\|_2 u$$

Now $\|p\|_2 = \Delta$, $\|p^*\|_2 = \Delta$ and $\|u\|_2 = 1$. So

$$\begin{aligned} \Delta^2 = p^T p &= (p^* + \|p-p^*\|_2 u)^T (p^* + \|p-p^*\|_2 u) \\ &= (p^*)^T p^* + 2\|p-p^*\|_2 u^T p^* + \|p-p^*\|_2^2 u^T u \\ &= \Delta^2 + \|p-p^*\|_2^2 \quad (\text{since } u^T p^* = 0) \end{aligned}$$

$$\Rightarrow \quad \|p - p^*\|_2 = 0$$

$$\Rightarrow \quad p - p^* = 0$$

$$\therefore \quad u = \frac{p - p^*}{\|p - p^*\|_2} = \frac{\underset{\sim}{0}}{0}$$ which is <u>not</u> well-defined.

The hint in how to complete the proof comes from N & W's claim that $\$$ is dense on the unit sphere.

So here's how I would comple the proof.

<u>If you have a better idea, let me know.</u>

We want to show $B + \lambda I$ is positive semi-definite. (We already know it is symmetric.)

So, we want to show

$$\frac{1}{2} \, x^T (B + \lambda I) x \geq 0 \quad \text{for all } x \in R^n \qquad (1)$$

Now, if $x = \underset{\sim}{0}$, (1) is obviously true. Therefore, we really only need to prove

$$\frac{1}{2} \, x^T (B + \lambda I) x \geq 0 \quad \text{for all } x \in R^n, \ x \neq \underset{\sim}{0} \qquad (2)$$

However, if $x \neq \underset{\sim}{0}$, then $\|x\|_2 > 0$. So, (2) is equivalent to

$$\frac{1}{2} \, \frac{x^T}{\|x\|_2} (B + \lambda I) \frac{x}{\|x\|_2} \geq 0 \quad \text{for all } x \in R^n, \ x \neq \underset{\sim}{0} \qquad (3)$$

Let $u = \frac{x}{\|x\|_2}$ and note that $\|u\|_2 = 1$.

So (3) is equivalent to

$$\tfrac{1}{2}\, u^T (B + \lambda I)\, u \geq 0 \qquad \text{for all } u \in R^n \text{ s.t. } \|u\|_2 = 1 \qquad (4)$$

So, if we can show (4) holds, then $B + \lambda I$ is symmetric positive-semidefinite. (Recall B is symmetric.)

We will prove (4) by contradiction.

Suppose (4) does not hold.

Then there is a $u \in R^n$ s.t. $\|u\|_2 = 1$ and

$$\tfrac{1}{2}\, u^T (B + \lambda I)\, u = \gamma < 0 \qquad (5)$$

Now suppose $u^T p^* \neq 0$.

Let $p = p^* + \alpha u$ (where α is yet to be determined)

Now note that $\|p^*\|_2 = \Delta$ $\|u\|_2 = 1$ and we want $\|p\|_2 = \Delta$.
So

$$\Delta^2 = p^T p = (p^* + \alpha u)^T (p^* + \alpha u)$$

$$= (p^*)^T p^* + 2\alpha\, u^T p^* + \alpha^2 u^T u$$

$$= \Delta^2 + 2\alpha\, u^T p^* + \alpha^2 \qquad \left(\text{since } (p^*)^T p = \Delta^2 \text{ and } u^T u = 1\right)$$

$$\therefore \quad 0 = 2\alpha\, u^T p^* + \alpha^2 = \alpha\,(2\, u^T p^* + \alpha)$$

Now $\alpha = 0$ does not give a useful solution,

since $\alpha = 0 \Rightarrow p = p^* \Rightarrow u = \frac{p - p^*}{\alpha} = \frac{0}{0}$

However, $\alpha = -2u^T p^* \neq 0$ does give a useful sol'n.

So, let $p = p^* + \alpha u = p^* - (2u^T p^*) u$

$$\Rightarrow \quad u = \frac{p - p^*}{\alpha} \quad \text{and} \quad \alpha \neq 0.$$

Therefore, we can write 5 as

$$\frac{1}{2} \left(\frac{p - p^*}{\alpha}\right)^T (B + \lambda I) \left(\frac{p - p^*}{\alpha}\right) = \gamma < 0 \qquad (6)$$

Multiplying 6 by $\alpha^2 > 0$, we get

$$\frac{1}{2} (p - p^*)^T (B + \lambda I)(p - p^*) = \alpha^2 \gamma < 0 \qquad (7)$$

Note that 7 contradicts 4.51

So, we just have the case $u^T p^* = 0$ to deal with.

In this case, consider

$$u_k = \sqrt{1 - 2^{-2k}}\, u + \frac{2^{-k}}{\Delta} p^*$$

Note $$u_k^T u_k = \left(\sqrt{1-2^{-2k}}\, u + \frac{2^{-k}}{\Delta} p^*\right)^T \left(\sqrt{1-2^{-2k}}\, u + \frac{2^{-k}}{\Delta} p^*\right)$$

$$= (1-2^{-2k})\, u^T u + \sqrt{1-2^{-2k}}\, \frac{2^{-k}}{\Delta} u^T p^* + \frac{2^{-2k}}{\Delta^2} (p^*)^T p^*$$

$$= 1 - 2^{-2k} + 2^{-2k} \qquad \left(\text{since } u^T p^* = 0 \text{ and } (p^*)^T p^* = \Delta^2\right)$$

$$= 1$$

$\therefore \quad \|u_k\|_2 = 1$

Also note that $u_k \to u$ as $k \to \infty$.
(That's where the hint that S' is dense on the unit sphere comes in.)

Now let $$p_k = p^* + \alpha_k u_k \qquad (\text{where } \alpha_k \text{ is yet to be determined})$$

We want $\|p_k\|_2 = \Delta$, We also have $\|p^*\|_2 = \Delta$, and $\|u_k\|_2 = 1$

$\therefore$ we want to choose α_k s.t.

$$\Delta^2 = p_k^T p_k = (p^* + \alpha_k u_k)^T (p^* + \alpha_k u_k)$$

$$= (p^*)^T p^* + 2\alpha_k u_k^T p^* + \alpha_k^2 u_k^T u_k$$

$$= \Delta^2 + 2\alpha_k u_k^T p^* + \alpha_k^2 \qquad \left(\text{since } (p^*)^T p^* = \Delta^2,\ u_k^T u_k = 1\right)$$

$$\Rightarrow 0 = 2\alpha_k u_k^T p^* + \alpha_k^2 = \alpha_k (2 u_k^T p^* + \alpha_k)$$

As before, $\alpha_K = 0$ does not give a useful solution.

So, let $\alpha_K = -2 u_K^T p^*$

$$= -2 \left(\sqrt{1-2^{-2K}}\, u + \frac{2^{-K}}{\Delta} p^* \right)^T p^*$$

$$= -2 \left(\sqrt{1-2^{-2K}}\, u^T p^* + \frac{2^{-K}}{\Delta} (p^*)^T p^* \right.$$

$$= -2 \, \frac{2^{-K}}{\Delta} \Delta^2 \qquad \left(\text{since } u^T p^* = 0 \text{ and } (p^*)^T p^* = \Delta^2 \right)$$

$$= -2^{-K+1} \Delta$$

$\therefore \quad p_K = p^* + \alpha_K u_K$

$$\Rightarrow \quad u_K = \frac{p_K - p^*}{\alpha_K}$$

Note $\|u_K\|_2 = 1$, $\|p_K\|_2 = \Delta$, $\|p^*\|_2 = \Delta$ and $\alpha_K \neq 0$.

Now consider

$$\frac{1}{2} u_K^T (B + \lambda I) u_K \qquad (8)$$

Since $u_K = \dfrac{p_K - p^*}{\alpha_K}$, we get from (8) that

$$\frac{1}{2} u_K^T (B + \lambda I) u_K = \frac{1}{2} \left(\frac{p_K - p^*}{\alpha_K} \right)^T (B + \lambda I) \frac{(p_K - p^*)}{\alpha_K}$$

$$= \frac{1}{\alpha_K^2} \frac{1}{2} (p_K - p^*)^T (B + \lambda I)(p_K - p^*) \geq 0 \qquad (9)$$

from (4.51) and $\alpha_k^2 > 0$.

On the other hand

$$\frac{1}{2} u_k^T (B_k + \lambda I) u_k$$

$$= \frac{1}{2} \left(\sqrt{1-2^{-2k}}\, u + \frac{2^{-k}}{\Delta} p^* \right)^T (B+\lambda I) \left(\sqrt{1-2^{-2k}}\, u + \frac{2^{-k}}{\Delta} p^* \right)$$

$$= \frac{1}{2} (1-2^{-2k})\, u^T (B+\lambda I)\, u$$

$$+ \sqrt{1-2^{-2k}}\, \frac{2^{-k}}{\Delta}\, u^T (B+\lambda I)\, p^*$$

$$+ \left(\frac{2^{-k}}{\Delta}\right)^2 (p^*)^T (B+\lambda I)\, p^*$$

$$\longrightarrow \frac{1}{2} u^T (B+\lambda I)\, u \quad \text{as} \quad k \to \infty$$

However, we assumed in (5) that

$$\frac{1}{2} u^T (B+\lambda I)\, u = \gamma < 0$$

$\therefore$ For k sufficiently large

$$\frac{1}{2} u_k^T (B+\lambda I)\, u_k < 0 \qquad (10)$$

But (10) contradicts (9).

$\therefore$ We cannot have a u s.t. $\|u\| = 1$ and
$\frac{1}{2} u^T (B+\lambda I)\, u < 0$.

Hence, $\frac{1}{2} u^T (B+\lambda I) u \geq 0$ for all $\|u\|=1$

$\Rightarrow \quad \frac{1}{2} x^T (B+\lambda I) x \geq 0$ for all $x \in R^n$, $x \neq 0$

But obviously $\frac{1}{2} x^T (B+\lambda I) x \geq 0$ for $x=0$ also

$\therefore \quad \frac{1}{2} x^T (B+\lambda I) x \geq 0$ for all $x \in R^n$

$\therefore \quad B+\lambda I$ is symmetric positive semi-definite.