Q&A Session 9 (Friday, April 3)

Other topics.

1. Preconditioned CG (N&W, p. 118)

Let $\hat{x} = Cx$ C nonsingular

$\Leftrightarrow x = C^{-1}\hat{x}$

$$\hat{\phi}(\hat{x}) = \phi(C^{-1}\hat{x}) = \frac{1}{2}\hat{x}^T(C^{-T}AC^{-1})\hat{x} - (C^{-T}b)^T\hat{x}$$

Sol'n $C^{-T}AC^{-1}\hat{x} = C^{-T}b$

Possible advantage: $K(C^{-T}AC^{-1}) \ll K(A)$

Eg. Suppose $A = LL^T$ (Cholesky factorization)

Let $C = L^T$

Then $(C^{-T})AC^{-1} = L^{-1}LL^TL^{-T} = I$

$\Rightarrow K(C^{-T}AC^{-1}) = 1$

Incomplete Cholesky Factorization $LL^T = A + E$
often $K(L^{-1}AL^{-T}) \ll K(A)$

Can implement Precondition CG quite efficiently

See Alg. 5.3 on p. 119 (One extra solve with M)

Matrix Free CG

CG needs matrix multiply Ap

Suppose $A = \nabla^2 f(x_k)$

Then $Ap = \nabla^2 f(x_k) p \simeq \dfrac{\nabla f(x_k + hp) - \nabla f(x_k)}{h}$

Can have problems with accuracy.

Can also use automatic differentiation to compute $\nabla^2 f(x_k) p$

Inexact Newton Method (N & W p. 165)

Newton step $\quad p_k = -\left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$

$$\iff \quad \nabla^2 f(x_k)\, p_k = -\nabla f(x_k)$$

$$\iff \quad \nabla^2 f(x_k)\, p_k + \nabla f(x_k) = 0$$

Inexact Newton methods – use an iterative method (e.g. CG) to compute a p_k that satisfies

$$\| \nabla^2 f(x_k)\, p_k + \nabla f(x_k) \| \le \eta_k \| \nabla f(x_k) \|$$

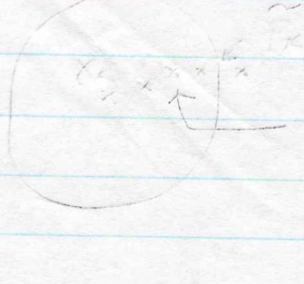
(a) $\eta_k \le \eta < 1$ get linear convergence (Thm 7.1)

(b) $\eta_k \to 0$ get super linear convergence

(c) $\eta_k = O(\| \nabla f(x_k) \|)$ get quadratic convergence

Trust-Region-Newton-CG (N & W, p. 170

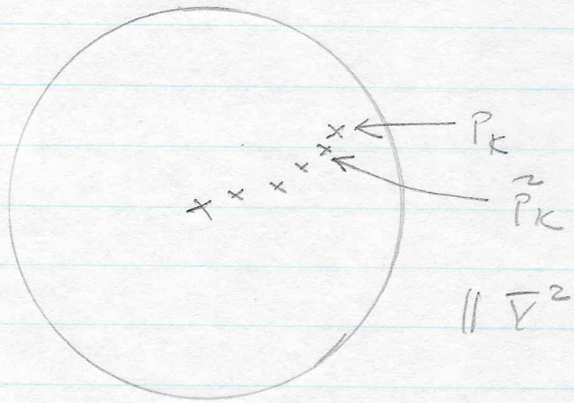
- Similar to Dog-Leg Stop when approx to p_k leaves the Trust Region



$\tilde{p}_k \approx p_k \quad \nabla^2 f(x_k)\, p_k = -\nabla f(x_k)$

j^{th} approx. to p_k generated by CG.

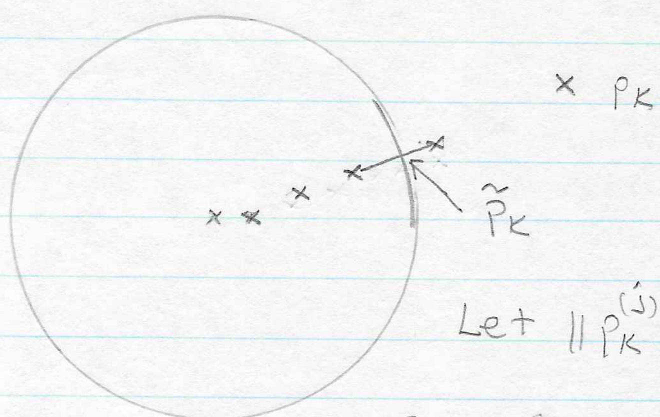
Case 1 $\quad \|p_K\| \le \Delta$



stop when

$$\|\nabla^2 f(x_K)\,\tilde{p}_K + \nabla f(x_K)\| \le \eta_K \|\nabla f(x_K)\|$$

Case 2 $\quad \|p_K\| > \Delta$



Let $\|p_K^{(j)}\| \le \Delta$ and $\|p_K^{(j+1)}\| > \Delta$

Consider line $p_K^{(j)} + \tau\,(p_K^{(j+1)} - p_K^{(j)})$

$\tau \in [0, 1]$

Choose τ^* s.t. $\tilde{p}_K = p_K^{(j)} + \tau^*(p_K^{(j+1)} - p_K^{(j)})$

satifies $\|\tilde{p}_K\| = \Delta$

Like dogleg method

(a) $\| p_K^{(j)} \|$ increasing with j

(b) $\phi(x_K + p_K^{(j)})$ decreasing with j.