Hard Case : p. 87 of N & W textbook

Assume $q_1^T g = 0$ q_1 eigenvector assoc. with λ_1

(Actually $q_i^T g = 0$ for q_i s.t. $\lambda_i = \lambda_1$.)

Then $p(\lambda) = -(B+\lambda I)^{-1} g$

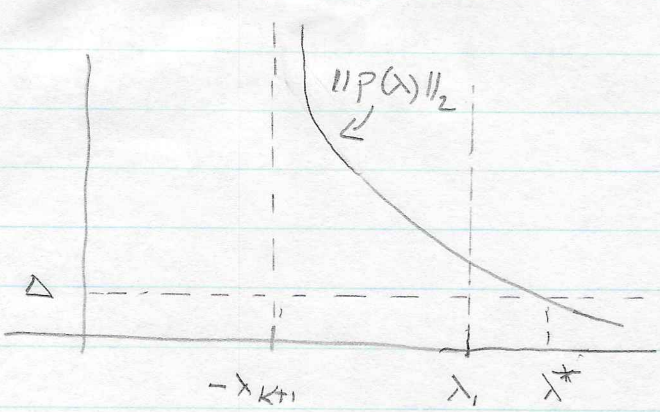
$$= -Q(\Lambda + \lambda I)^{-1} Q^T g$$

see p. 64 of N & W. textbook

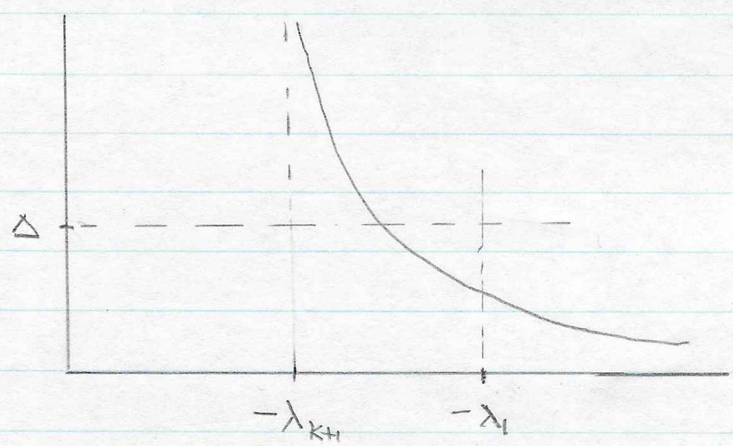
$$= -\sum_{j=1}^{n} \frac{q_j^T g}{\lambda_j + \lambda} q_j$$

$$= -\sum_{j=k+1}^{n} \frac{q_j^T g}{\lambda_j + \lambda} q_j$$

(assuming $\lambda_i = \lambda_1$ for $i = 1, 2, \dots, k$)



(a)



(b)

Case (a) just like easy case. Just find λ^+ s.t. $\|p(\lambda^+)\|_2 = \Delta$

Case (b) hard case: no $\lambda \geq -\lambda_1$ s.t. $\|p(\lambda)\|_2 = \Delta$.

However, let $\lambda = -\lambda_1$ and $z = q_1$

Recall $Bq_1 = \lambda_1 q_1 \Rightarrow (B - \lambda_1 I) q_1 = 0$

So consider $p(-\lambda_1;\tau) = \sum_{j=k+1}^{n} \frac{q_j^T g}{\lambda_j - \lambda_1} q_j + \tau q_1$

book uses z in (4.45)

Recall $(B + \lambda I)\, p(\lambda) = -g$ (see page 1)

$\therefore \; (B - \lambda_1 I)\, p(-\lambda_1) = -g$

$\therefore \; (B - \lambda_1 I)(p(-\lambda_1;\tau)) = -g + \tau \underbrace{(B - \lambda_1 I) q_1}_{=0}$

$= -g$ (this is (4.8a))

Now $\|p(-\lambda_1;\tau)\|_2^2 = \|p(-\lambda_1)\|_2^2 + \tau^2$

recall (Hard Case) $\|p(-\lambda_1)\|_2 < \Delta$

$\therefore$ Just need to find τ^* s.t. $(\tau^*)^2 = \Delta^2 - \|p(-\lambda_1)\|_2^2 > 0$

$$\tau^* = \sqrt{\Delta^2 - \|p(-\lambda_1)\|_2^2}$$

Proof of Theorem 4.1   N+W p. 90

Assume there is a p^* satisfying $\|p^*\| \leq \Delta$
and a $\lambda \geq 0$ s.t. (4.8a), (4.8b), (4.8c) are satisfied

Consider $\hat{m}(p) = g^T p + \frac{1}{2} p^T (B + \lambda I) p$

(4.8a) $\Rightarrow (B + \lambda I) p^* = -g \Rightarrow (B + \lambda) p^* + g = 0$

$\Rightarrow \nabla \hat{m}(p^*) = 0$

(4.8c) $\Rightarrow B + \lambda I$ is positive semi-definite

$\Rightarrow p^*$ is a minimizer of $\hat{m}(p)$
i.e. $\hat{m}(p) \geq \hat{m}(p^*)$

However, $\hat{m}(p) = m(p) + \frac{\lambda}{2} p^T p$

$\therefore \hat{m}(p) \geq \hat{m}(p^*) \Rightarrow m(p) + \frac{\lambda}{2} p^T p \geq m(p^*) + \frac{\lambda}{2} (p^*)^T p$

$\Rightarrow m(p) \geq m(p^*) + \frac{\lambda}{2} ((p^*)^T p^* - p^T p)$ ①

(4.8b) $\Rightarrow \lambda (\Delta - \|p^*\|) = 0$

$\therefore$ ① $\Rightarrow m(p) \geq m(p^*) + \frac{\lambda}{2} (\Delta^2 - p^T p)$ ②

$\Rightarrow m(p) \geq m(p^*)$ for all $\|p\| \leq \Delta$

$\therefore p^*$ is the minimizer of (4.7)

Converse: Suppose p^* is a solution of (4.7).

Want to show p^* satisfies (4.8a), (4.8b), (4.8c).

Simple case: assume $\|p^*\| < \Delta$.

$\Rightarrow$ p^* is an unconstrained minimizer of $m(p)$

$\Rightarrow$ $\nabla m(p^*) = Bp^* + g = 0$ and $\nabla^2 m(p^*)$ is positive -semidefinite

$\therefore$ p^* satisfies (4.8a), 4.8b), (4.8c) with $\lambda = 0$

$\therefore$ Assume $\|p^*\| = \Delta$

So (4.8b) is satisfied.

(4.7) is equivalent to $\min m(p)$ s.t. $\|p\| = \Delta$ ①

Form the Lagrangian for ①

$$\mathcal{L}(p, \lambda) = m(p) + \frac{\lambda}{2}(p^T p - \Delta^2)$$

The solution p^* of ① must satisfy

$$0 = \nabla_p \mathcal{L}(p^*, \lambda) = Bp^* + g + \lambda p^*$$

$$\Rightarrow \quad (B + \lambda I)p^* = -g$$ (4.8a) is satisfied

$\therefore$ (4.8a) is satisfied.

Now note $m(p) \geq m(p^*)$

for all p s.t. $\|p\|_2 = \Delta \Rightarrow p^T p = \Delta^2$

also $(p^*)^T p^* = \Delta^2$

$\therefore$ $$m(p) \geq m(p^*) + \frac{\lambda}{2}\left((p^*)^T p^* - p^T p\right) \quad (3)$$

Use $(B+\lambda I)\, p^* = -g$ and re-arrange (3) to get

$$\frac{1}{2}\,(p-p^*)^T\,(B+\lambda I)(p-p^*) \geq 0 \quad (4)$$

Actually want

$$u^T (B+\lambda I)\, u \geq 0 \quad \text{for all } u \in R^n \quad (5)$$

Suppose (5) not true. Then

There is a u^* s.t. $\|u^*\| = 1$ and

$$\frac{1}{2}(u^*)^T (B+\lambda I)\, u^* = \min_{\|u\|=1} \frac{1}{2} u^T (B+\lambda I)\, u = \alpha < 0 \quad (6)$$

Now claim that the set

$$S = \left\{ w : w = \frac{p-p^*}{\|p-p^*\|},\ \|p\| = \Delta \right\}$$

is dense on the unit sphere.

Now (4) $\Rightarrow$ $\frac{1}{2}\, w\,(B+\lambda I)\, w \geq 0 \ \forall\ w \in S$.

Also, $\frac{1}{2} u^T (B+\lambda I) u$ is a continuous function

So, for any $\varepsilon > 0$ can choose a $\delta > 0$ s.t.

$$\left| \frac{1}{2} u^T (B+\lambda I) u - \frac{1}{2} w^T (B+\lambda I) w \right| < \varepsilon$$

for $|u - w| < \delta$

If S is dense on the unit sphere, can choose a $w^* \in S$ s.t.

$$\| w^* - u^* \| < \delta$$

$$\Rightarrow \quad \left| \frac{1}{2} (u^*)^T (B+\lambda I) u^* - \frac{1}{2} (w^*)^T (B+\lambda I) w^* \right| < \varepsilon$$

But this contradicts $\quad \frac{1}{2} (w^*)^T (B+\lambda I) w^* \geq 0$

and $\quad \frac{1}{2} (u^*)^T (B+\lambda I) u^* = \alpha < 0$

Question: is S dense on the unit sphere?

I.e. for any u s.t. $\|u\| = 1$ and any $\varepsilon > 0$

is there $w \in S$ s.t. $\| u - w \| < \varepsilon$?