Duration — 50 minutes

Do **NOT** turn this page over until you are **TOLD** to start.

Answer **ALL** Questions in the Test Booklets Provided

This is a **closed-book test**: no books, no notes, no calculators, no phones, no tablets, no computers (of any kind) allowed.

Please fill-in **ALL** the information requested on the front cover of **EACH** test booklet that you use.

The test consists of 4 pages, including this one. Make sure you have all 4.

The test consists of 3 questions. **Answer all 3 questions**. The mark for each question is listed at the start of the question. Do the questions that you feel are easiest first.

The test was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity**.

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.

You may find the following definitions useful.
(Assume $h \in \mathbb{R}$ is a small positive stepsize.)

The *shift* operator:	$\mathcal{E}z(x) = z(x+h)$
The *forward difference* operator:	$\Delta_+ z(x) = z(x+h) - z(x)$
The *backward difference* operator:	$\Delta_- z(x) = z(x) - z(x-h)$
The *central difference* operator:	$\Delta_0 z(x) = z(x+h/2) - z(x-h/2)$
The *averaging* operator:	$\Upsilon_0 z(x) = \frac{1}{2}\Big(z(x+h/2) + z(x-h/2)\Big)$
The *derivative* operator:	$Dz(x) = z'(x)$

1. [5 marks]

   Show that formally

$$\Upsilon_0 \Delta_0 = \sum_{i=0}^{\infty} \frac{(hD)^{2i+1}}{(2i+1)!}$$

   Hence

$$\frac{1}{h}\Upsilon_0 \Delta_0 = D + \mathcal{O}(h^2)$$

2. [10 marks: 5 marks for part (a) and 5 marks for part (b)]

Consider the two-point boundary value problem

$$\begin{aligned} -y''(x) + y'(x) + y(x) &= f(x), \qquad x \in (0,1) \\ y(0) = \alpha, \qquad y(1) &= \beta \end{aligned} \tag{1}$$

Suppose that we discretize (1) as follows:

$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{y_{i+1} - y_{i-1}}{2h} + y_i = f_i \qquad \text{for } i = 1, 2, \ldots, m \tag{2}$$

where m is a positive integer, $h = \frac{1}{m+1}$, $y_0 = y(0)$, $y_i \approx y(ih)$ for $i = 1, 2, \ldots, m$, $y_{m+1} = y(1)$ and $f_i = f(ih)$ for $i = 1, 2, \ldots, m$.

(a) To find the unknowns y_i, $i = 1, 2, \ldots, m$, in (2), we usually form a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ and solve $\mathbf{A}\mathbf{y} = \mathbf{b}$ for the vector $\mathbf{y} = [y_1, y_2, \ldots, y_m]^T$.

i. What are the values of the elements $\mathbf{A}_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$, of the matrix $\mathbf{A}$?

ii. What are the values of the elements $\mathbf{b}_i$, $i = 1, 2, \ldots, m$, of the vector $\mathbf{b}$?

(b) What is the order of consistency of the numerical method (2)?

Justify your answer.

3. [12 marks: 2 marks for part (a); 5 marks for part (b); 5 marks for part (c)]

Consider the two-point boundary value problem

$$\begin{aligned} -\Big((1+x)y'(x)\Big)' + y(x) = 1, \qquad & x \in (0,1) \\ y(0) = -1 \quad \text{and} \quad y(1) = 1 & \end{aligned} \tag{3}$$

For the Ritz-Galerkin method, we approximate the solution $y(x)$ of (3) by

$$y_m(x) = \phi_0(x) + \sum_{k=1}^{m} \gamma_k \, \phi_k(x) \tag{4}$$

where m is a positive integer, $\phi_0(x)$ is a function that satisfies the boundary conditions (i.e., $\phi_0(0) = -1$ and $\phi_0(1) = 1$) and the $\phi_k(x)$, $k = 1, 2, \ldots, m$, are basis functions that satisfy zero boundary conditions (i.e., $\phi_k(0) = 0$ and $\phi_k(1) = 0$, for $k = 1, 2, \ldots, m$). In this question, assume that the $\phi_k(x)$ are the piecewise linear hat (i.e., chapeau) functions

$$\phi_k(x) = \begin{cases} \frac{x - x_{k-1}}{h} & \text{for } x \in [x_{k-1}, x_k] \\ \frac{x_{k+1} - x}{h} & \text{for } x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, \ldots, m$ on a uniform grid $\{x_k = kh : k = 0, 1, 2, \ldots, m+1\}$, where $h = 1/(m+1)$. This gives rise to a system of linear equations of the form

$$\mathbf{A}\gamma = \mathbf{b} \tag{5}$$

where $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_m]^T$ is the vector of coefficients of the piecewise linear approximation $y_m(x)$ (see (4)) to the solution $y(x)$ of (3).

(a) What is a good choice for the function $\phi_0(x)$ for the two-point boundary value problem (3) described above?

Justify your answer.

(b) For the Ritz-Galerkin method for the two-point boundary value problem (3) described above, give the values of $\mathbf{A}_{k,l}$ for $k = 1, \ldots, m$ and $l = 1, \ldots, m$, where $\mathbf{A}_{k,l}$ is element (k, l) of the matrix $\mathbf{A}$ in equation (5).

(c) For the Ritz-Galerkin method for the two-point boundary value problem (3) described above, give the values of $\mathbf{b}_k$ for $k = 1, \ldots, m$, where $\mathbf{b}_k$ is element k of the vector $\mathbf{b}$ in equation (5).

Advice: in parts (b) and (c), you need to compute the values of several integrals. If you can't compute the value of an integral, at least specify the integral the value of which you want to compute.