University of Toronto

Faculty of Arts and Science

April 2017 Examinations

CSC 446/2310 H1S
Computational Methods for Partial Differential Equations

Duration — 3 hours

No Aids Allowed

Answer ALL Questions

Do **NOT** turn this page over until you are **TOLD** to start.

Write your answers in the exam booklets provided.

Please fill-in **ALL** the information requested on the front cover of **EACH** exam booklet that you use.

The exam consists of 7 pages, including this one. Make sure you have all 7 pages.

The exam consists of 4 questions. **Answer all 4 questions**. The mark for each question is listed at the start of the question. Do the questions that you feel are easiest first.

The exam was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity**.

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.

You may find the following definitions useful.

The *shift* operator:	$\mathcal{E} z(x) = z(x+h)$
The *forward difference* operator:	$\Delta_+ z(x) = z(x+h) - z(x)$
The *backward difference* operator:	$\Delta_- z(x) = z(x) - z(x-h)$
The *central difference* operator:	$\Delta_0 z(x) = z(x+h/2) - z(x-h/2)$
The *averaging* operator:	$\Upsilon_0 z(x) = \frac{1}{2}\big(z(x+h/2) + z(x-h/2)\big)$
The *differentiation* operator:	$D z(x) = z'(x)$

1. [12 marks: 5 marks for part (a), 5 marks for part (b) and 2 marks for part (c)]

The operators above can be extended to functions $u(x,y)$ in two spatial dimensions. For example, let

$$\begin{aligned}
\Delta_{+,x} u(x,y) &= u(x+h_x,y) - u(x,y) \\
\Delta_{+,y} u(x,y) &= u(x,y+h_y) - u(x,y) \\
\Delta_{-,x} u(x,y) &= u(x,y) - u(x-h_x,y) \\
\Delta_{-,y} u(x,y) &= u(x,y) - u(x,y-h_y) \\
D_x u(x,y) &= \frac{\partial u(x,y)}{\partial x} \\
D_y u(x,y) &= \frac{\partial u(x,y)}{\partial y}
\end{aligned}$$

where $h_x \in \mathbb{R}$, $h_x > 0$, $h_y \in \mathbb{R}$ and $h_y > 0$. In this question, do <u>NOT</u> assume $h_x = h_y$.

Consider the finite-difference approximation

$$\alpha \, \frac{\Delta_{+,x}\Delta_{+,y} u(x,y)}{h_x h_y} + (1-\alpha) \, \frac{\Delta_{-,x}\Delta_{-,y} u(x,y)}{h_x h_y} \tag{1}$$

to the partial derivative

$$\frac{\partial^2 u(x,y)}{\partial x \, \partial y} \tag{2}$$

where $\alpha \in \mathbb{R}$ and $\alpha \in [0,1]$.

(a) Show the *computational stencil* associated with the finite-difference approximation (1).

(b) Show that, for any $\alpha \in [0,1]$, the finite-difference approximation (1) is at least a first-order approximation to the partial derivative (2).

Include enough terms in your error expansion so that you can answer part (c) below.

(c) For what values of α is the finite-difference approximation (1) a second-order approximation to the partial derivative (2)?

2. [10 marks: 5 marks for each part]

Consider the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y) \qquad \text{for } (x,y) \in \Omega_L \tag{3}$$

in two-dimensions with Dirichlet boundary conditions

$$u(x,y) = g(x,y) \qquad \text{for } (x,y) \in \partial\Omega_L \tag{4}$$

where the domain Ω_L is the *L-shaped* region

$$\Omega_L = R_1 \cup R_2$$

and R_1 and R_2 are the rectangles

$$\begin{aligned} R_1 &= \{(x,y) : -1 < x < 1 \text{ and } -1 < y < 0\} \\ R_2 &= \{(x,y) : -1 < x < 0 \text{ and } 0 \le y < 1\} \end{aligned}$$

Another way of describing Ω_L is that it is the square

$$S_1 = \{(x,y) : -1 < x < 1 \text{ and } -1 < y < 1\}$$

with the smaller square

$$S_2 = \{(x,y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$$

removed. That is,

$$\Omega_L = \{(x,y) : (x,y) \in S_1 \text{ and } (x,y) \notin S_2\}$$

For any integer $N \ge 1$, let $h = 1/(N+1)$ and consider the discretization

$$\begin{aligned} x_i &= -1 + i\,h \quad \text{for } i = 0, 1, \ldots, 2(N+1) \\ y_j &= -1 + j\,h \quad \text{for } j = 0, 1, \ldots, 2(N+1) \end{aligned}$$

Note $(x_i, y_j) \in \Omega_L$ if either

(a) $i \in \{1, 2, \ldots, 2N+1\}$ and $j \in \{1, 2, \ldots, N\}$ or

(b) $i \in \{1, 2, \ldots, N\}$ and $j \in \{N+1, N+2, \ldots, 2N+1\}$.

Using this discretization and the 5-point approximation to the Laplacian

$$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

where $u_{i,j} \approx u(x_i, y_j)$, you can construct a system of linear equations

$$A\hat{u} = \hat{f} + \hat{g} \tag{5}$$

where $\hat{u}$ is a vectorized version of $\{u_{i,j} : (x_i, y_j) \in \Omega_L\}$, $\hat{f}$ is a vectorized version of the function $f(x,y)$ on the right side of the Poisson equation (3), and $\hat{g}$ is a vector containing the boundary conditions corresponding to (4).

(a) Describe how to initialize the matrix A and the vectors $\hat{f}$ and $\hat{g}$ in (5).

Your description should be detailed enough so that a programmer, who doesn't know anything about numerical methods for PDEs, can write a program to initialize the matrix A and the vectors $\hat{f}$ and $\hat{g}$.

Suggestion: you may find it useful to use MatLab pseudo-code. (Your MatLab code does not have to be syntactically correct; it just has to give a clear idea how to initialize the matrix A and the vectors $\hat{f}$ and $\hat{g}$.)

(b) Show that the matrix A in (5) is nonsingular.

3. [15 marks: 5 marks for each part]

Consider the two-point boundary value problem (BVP)

$$\begin{aligned} -(a(x)\, y'(x))' + b(x)\, y(x) &= f(x) \qquad \text{for } x \in (0,1) \\ y'(0) &= \alpha \\ y'(1) &= \beta \end{aligned} \tag{6}$$

where $a(x) > 0$ for all $x \in [0,1]$ and $b(x) > 0$ for all $x \in [0,1]$.

Note: we normally assume $b(x) \geq 0$ for all $x \in [0,1]$, but in this question assume $b(x) > 0$ for all $x \in [0,1]$.

(a) How can you transform the BVP (6) to a BVP with homogeneous (i.e., zero) boundary conditions of the form

$$\begin{aligned} -(a(x)\, z'(x))' + b(x)\, z(x) &= g(x) \qquad \text{for } x \in (0,1) \\ z'(0) &= 0 \\ z'(1) &= 0 \end{aligned} \tag{7}$$

In particular, describe how the solution $z(x)$ of (7) relates to the solution $y(x)$ of (6) and how $g(x)$ in (7) relates to $f(x)$ in (6).

(b) Consider the grid

$$x_{-1} < 0 = x_0 < x_1 < x_2 < \cdots < x_n = 1 < x_{n+1} \tag{8}$$

Do <u>NOT</u> assume that the grid spacing is uniform. (That is, do <u>NOT</u> assume $x_{i+1} - x_i = x_{j+1} - x_j$ for $i \neq j$.) Given the grid (8), we can approximate the solution z to the BVP (7) by

$$z_n(x) = \sum_{j=0}^{n} \gamma_j\, \varphi_j(x) \tag{9}$$

where the coefficients γ_j, $j = 0, 1, \ldots, n$, are real constants and the piecewise linear hat (i.e., chapeau) basis functions $\varphi_j(x)$, $j = 0, 1, \ldots, n$, are defined by

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & \text{for } x \in [x_{j-1}, x_j] \\ \frac{x_{j+1} - x}{x_{j+1} - x_j} & \text{for } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

We can find the coefficients γ_j, $j = 0, 1, \ldots, n$, in (9) by converting (7) to its *weak form* and solving the associated *Galerkin* equations. Show that this leads to a linear system of algebraic equations of the form

$$A\vec{\gamma} = \vec{b} \tag{10}$$

where $\vec{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)^T$ and the (i, j) element of the $(n+1) \times (n+1)$ matrix A is

$$A_{i,j} = \int_0^1 a(x)\, \varphi_i'(x)\, \varphi_j'(x)\, dx + \int_0^1 b(x)\, \varphi_i(x)\, \varphi_j(x)\, dx \tag{11}$$

for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, n$. Also, for $i = 0, 1, \ldots, n$, give an equation for b_i, which is element i of the vector $\vec{b}$ in (10).

(c) Show that the matrix A in (10) with elements $A_{i,j}$ given by (11) is symmetric positive-definite.

4. [10 marks: 5 marks for each part]

I mentioned a few times in class that there are two equivalent definitions of an $n \times n$ matrix A being *reducible* (or equivalently *irreducible*).

Definition 1: An $n \times n$ matrix A is *reducible* if there are two nonempty sets I_1 and I_2 such that $I_1 \cup I_2 = \{1, 2, \ldots, n\}$, $I_1 \cap I_2 = \emptyset$ and $A_{i,j} = 0$ for all $i \in I_1$ and $j \in I_2$. Otherwise, the matrix A is *irreducible.*

Definition 2: An $n \times n$ matrix A is *irreducible* if for every $i \in \{1, 2, \ldots, n\}$ and every $j \in \{1, 2, \ldots, n\}$ there is a sequence $i = k_1, k_2, \ldots, k_s = j$ with $s \geq 2$ such that $A_{k_r,k_{r+1}} \neq 0$ for $r = 1, 2, \ldots, s - 1$. Otherwise, the matrix A is *reducible.*

Show that these two definitions are equivalent. That is, show that

(a) If an $n \times n$ matrix A is *reducible* according to Definition 1, then it is *reducible* according to Definition 2.

(b) If an $n \times n$ matrix A is *reducible* according to Definition 2, then it is *reducible* according to Definition 1.

Total Marks = 47

Total Pages = 7