This assignment is due at the start of the lecture on Thursday, 28 November 2019.

For the questions that require you to write a MatLab program, hand-in the program and its output as well as any written answers requested in the question. Your program and its output, as well as your written answers, will be marked. Your program should conform to the usual CS standards for comments, good programming style, etc.

When first learning to program in MatLab, students often produce long, messy output, but you should be an experienced MatLab programmer now. So, try to format the output from your programs so that it is easy for your TA to read and to understand your results. If needed, you might find it helpful to read "A short description of fprintf" on the course webpage `http://www.cs.toronto.edu/~krj/courses/336/`. Marks will be awarded for well-formatted, easy-to-read output.

Also, your TAs will appreciate your using a word processor to write the answers to questions (or parts of questions) that do not require a program. If you do write those answers by hand, make sure that they are easy to read.

1. [15 marks]

   Do question 3 on last year's final exam, which you can find on the course webpage `http://www.cs.toronto.edu/~krj/courses/336/`.

2. [15 marks: 5 marks for each part]

   We talked in class about using a vector $p = [p_1, p_2, \ldots, p_{n-1}]$ to represent the $n \times n$ elementary permutation matrices $P_1, P_2, \ldots, P_{n-1}$ that we use in Gaussian Elimination with row-partial-pivoting (i.e., the type of pivoting that we discussed in class and is described in Section 2.4.5 of your textbook). Recall that, in this notation, if $y = P_k\, x$, then y is the same as x, except that the elements x_k and x_j are interchanged, for some $j \geq k$. That is,

   $$\begin{aligned} y_i &= x_i \quad \text{if } i \in \{1, 2, \ldots, n\} \text{ and } i \notin \{k, j\} \\ y_k &= x_j \\ y_j &= x_k \end{aligned}$$

   If $j = k$, then this interchange effectively does nothing and so $P_k = I$ and $y = x$. However, if $j > k$, then $y \neq x$, unless (by chance) $x_k = x_j$.

   We represented the elementary permutation matrix P_k in the vector p by setting $p_k = j$, since all you really need to know about P_k is that it is a permutation matrix that does nothing except interchange elements k and j, where $j = p_k$, when you perform the matrix-vector multiply $P_k\, x$.

For example, the vector $p = [3, 2, 4]$ represents the three 4×4 elementary permutation matrices

$$P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_2 = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

When we compute the LU factorization of a matrix using row-partial-pivoting, we also need to represent the $n \times n$ permutation matrix P, where

$$P = P_{n-1} P_{n-2} \cdots P_2 P_1$$

One way to represent an $n \times n$ permutation matrix P is with a n-vector q for which $q_i = j$ if and only if $P_{ij} = 1$ (i.e., row i of P has a 1 in column j). Since a permutation matrix has exactly one 1 in each row (and all other elements are zero), this representation is very effective. For example, the 5×5 permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

can be represented by the vector $q = [3, 4, 5, 1, 2]$.

(a) Write a MatLab function

```
function y = perm_a(p,x)
```

that takes as arguments

- an $(n-1)$-vector p that represents the $n \times n$ elementary permutation matrices $P_1, P_2, \ldots, P_{n-1}$ as described above
- an n-vector x

and computes the n-vector y satisfying

$$y = P_{n-1} P_{n-2} \cdots P_2 P_1 x$$

Your function should not compute the matrices $P_1, P_2, \ldots, P_{n-1}$; it should use p only to compute y from x. In particular, it should compute y in time proportional to n.

(b) Write a MatLab function

```
function q = perm_b(p)
```

that takes as its argument

- an $(n-1)$-vector p that represents the $n \times n$ elementary permutation matrices $P_1, P_2, \ldots, P_{n-1}$ as described above, and

and computes the n-vector q that represents the permutation matrix P satisfying

$$P = P_{n-1} P_{n-2} \cdots P_2 P_1$$

Your function should not compute the matrices $P_1, P_2, \ldots, P_{n-1}$ or P; it should use p only to compute q. In particular, it should compute q in time proportional to n.

(c) Write a MatLab function

```
function y = perm_c(q,x)
```

that takes as arguments

- an n-vector q that represents the $n \times n$ permutation matrix P as described above, and
- an n-vector x

and computes the n-vector y satisfying

$$y = Px$$

Your function should not compute the matrix P; it should use q only to compute y from x. In particular, it should compute y in time proportional to n.

To test your functions `perm_a`, `perm_b` and `perm_c`, let

$$\texttt{p} = [5, 4, 9, 10, 6, 8, 10, 9, 10]$$

and

$$x = [1:10]'$$

(I.e., p is row vector with 9 elements and x is a column vector with 10 elements.)

Compute

```
y1 = perm_a(p,x)
q  = perm_b(p)
y2 = perm_c(q,x)
```

Hand in your MatLab program, the functions `perm_a`, `perm_b` and `perm_c` and their output for the test case above.

3. [10 marks: 5 marks each for parts (a) and (b)]

   Do question 5.2 listed under Computer Problems

   - on page 250 of the old McGraw-Hill version of Heath's textbook, and
   - on page 249 of new SIAM version of Heath's textbook.

   You can find a copy of pages 250 and 251 of the old McGraw-Hill version of Heath's textbook on the course webpage `http://www.cs.toronto.edu/~krj/courses/336/`.

4. [10 marks: 5 marks each for parts (a) and (b)]

   Consider the equation

   $$f(x) = x^2 - 2$$

   The roots of $f(x)$ are obviously $\pm\sqrt{2}$.

   (a) Write a MatLab program that starts with the initial guess $x_0 = 1$ and uses Newton's method to compute the positive root of $f(x)$. Your program should print a table of values with the header

n	x_n	$x_n - \sqrt{2}$

   You may not be able to do subscripts properly in MatLab; if so, you can use $x(n)$ instead of x_n. Similarly for $\sqrt{2}$. Also, add additional spaces between the n, x_n and $x_n - \sqrt{2}$ as necessary to make the table "look nice".

   Your program should then print six lines, one line for each of $n = 0, 1, 2, \ldots, 5$. Each line should contain the values for n, x_n and $x_n - \sqrt{2}$ for the n associated with that line. Print n as an integer and both x_n and $x_n - \sqrt{2}$ in the fprintf 20.15f format (or the equivalent if you use another output function).

   (b) Write a MatLab program that starts with the initial guesses $x_0 = 1$ and $x_1 = 2$ and uses the secant method to compute the positive root of $f(x)$. Your program should print a table of values with the header

n	x_n	$x_n - \sqrt{2}$

   You may not be able to do subscripts properly in MatLab; if so, you can use $x(n)$ instead of x_n. Similarly for $\sqrt{2}$. Also, add additional spaces between the n, x_n and $x_n - \sqrt{2}$ as necessary to make the table "look nice".

   Your program should then print eight lines, one line for each of $n = 0, 1, 2, \ldots, 7$. Each line should contain the values for n, x_n and $x_n - \sqrt{2}$ for the n associated with that line. Print n as an integer and both x_n and $x_n - \sqrt{2}$ in the fprintf 20.15f format (or the equivalent if you use another output function).

   If you need documentation for fprintf, see the course webpage `http://www.cs.toronto.edu/~krj/courses/336/`.

5. [10 marks]

Do question 5.14 listed under Computer Problems

- on page 251 of the old McGraw-Hill version of Heath's textbook, and
- on page 250 of new SIAM version of Heath's textbook.

You can find a copy of pages 250 and 251 of the old McGraw-Hill version of Heath's textbook on the course webpage `http://www.cs.toronto.edu/~krj/courses/336/`.

The "zero finder" that you should use for this question is the MatLab function fzero. In particular, use the version of fzero described under "Root Starting From an Interval" on the webpage `http://www.mathworks.com/help/matlab/ref/fzero.html`. You may also find reading "help fzero" in MatLab helpful.

6. [15 marks]

Do question 4 on last year's final exam, which you can find on the course webpage `http://www.cs.toronto.edu/~krj/courses/336/`.

On the exam last year, several students proved the results you are required to prove in this question for $f(x) = x^2 - 1$ only. The function $f(x) = x^2 - 1$ is meant to serve as an example only of a function $f(x)$ for which

(i) $f''(x)$ exists and is continuous for all $x \in \mathbb{R}$,

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, and

(iii) there is a point $\hat{x} \in \mathbb{R}$ for which $f'(\hat{x}) = 0$ and $f(\hat{x}) < 0$.

You are meant to prove the results specified in parts (a), (b) and (c) of this question for all functions $f(x)$ that satisfy points (i), (ii) and (iii) above. Do not prove the results only for $f(x) = x^2 - 1$.