$$A = \begin{pmatrix} -2 & 10 & 1 \\ 1 & -4 & 2 \\ 4 & -8 & 4 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

We want to find a solutionb for $Ax = b$
4 is the highest value in the first column and is in the third row. It needs to be brought to the top left. This will cause the multipliers to always be less than 1.

We will use the permutation matrix $P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

This will switch the first and third rows

$A^* = P_1 A = \begin{pmatrix} 4 & -8 & 4 \\ 1 & -4 & 2 \\ -2 & 10 & 1 \end{pmatrix}$

Now we will compute M_1
$M_1 = I - m_1 e_1^T$

$$m_1 = \begin{pmatrix} 0 \\ \\ \frac{A^*_{21}}{A^*_{11}} \\ \\ \frac{A^*_{31}}{A11^*} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \frac{1}{4} \\ \\ \frac{-2}{4} \end{pmatrix}, \quad m_1 e_1^T = \begin{pmatrix} 0 \\ \\ \frac{1}{4} \\ \\ \frac{-2}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \\ \frac{1}{4} & 0 & 0 \\ \\ \frac{-1}{2} & 0 & 0 \end{pmatrix}$$

$$I - m_1 e_1^T = \begin{pmatrix} 1 & 0 & 0 \\ \\ 0 & 1 & 0 \\ \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \\ \frac{1}{4} & 0 & 0 \\ \\ \frac{-1}{2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \\ \frac{-1}{4} & 1 & 0 \\ \\ \frac{1}{2} & 0 & 1 \end{pmatrix} = M_1$$

Now we compute $M_1P_1A = M_1A^*$

$$\hat{A} = M_1A^* = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -8 & 4 \\ 1 & -4 & 2 \\ -2 & 10 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -8 & 4 \\ 0 & -2 & 1 \\ 0 & 6 & 3 \end{pmatrix}$$

Now we observe that between the second and third rows, the largest value in the second column is 6, which is in the third row. We need to switch it to the second row. So we will use a permutation matrix P_2

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus

$$\hat{A}^* = P_2\hat{A} = \begin{pmatrix} 4 & -8 & 4 \\ 0 & 6 & 3 \\ 0 & -2 & 1 \end{pmatrix}$$

Now we compute M_2

$M_2 = I - m_2e_2^T$

$$m2 = \begin{pmatrix} 0 \\ 0 \\ \frac{\hat{A^*_{32}}}{\hat{A^*_{22}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-2}{6} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{3} \end{pmatrix}$$

$$m_2 e_2^T = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1}{3} & 0 \end{pmatrix}$$

$$M_2 = I - m_2 e_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1}{3} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$$

Finally we will compute $U = M_2 \hat{A}^* = M_2 P_2 M_1 P_1 A$

$$U = M_2 \hat{A}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 4 & -8 & 4 \\ 0 & 6 & 3 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -8 & 4 \\ 0 & 6 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Now we want to compute something similar for b

$$b_2 = M_2 P_2 M_1 P_1 b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{11}{4} \\ \frac{9}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{9}{2} \\ \frac{11}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{9}{2} \\ \frac{17}{4} \end{pmatrix}$$

So now we have the equation $Ux = b_2$

$$\begin{pmatrix} 4 & -8 & 4 \\ 0 & 6 & 3 \\ 0 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ \frac{9}{2} \\ \frac{17}{4} \end{pmatrix}$$

We will solve the following equations

$4x_1 - 8x_2 + 4x_3 = 1$

$6x_2 + 3x_3 = \frac{9}{2}$

$2x_3 = \frac{17}{4}$

$x_3 = \frac{\frac{17}{4}}{2} = \frac{17}{8}$

$x_2 = \frac{\frac{9}{2} - 3x_3}{6} = \frac{\frac{9}{2} - 3\frac{17}{8}}{6} = \frac{\frac{36}{8} - \frac{51}{8}}{6} = \frac{-\frac{15}{8}}{6} = -\frac{15}{48} = -\frac{5}{16}$

$x_1 = \frac{1 + 8x_2 - 4x_3}{4} = \frac{1 - 8\frac{5}{16} - 4\frac{17}{8}}{4} = \frac{\frac{2}{2} - \frac{5}{2} - \frac{17}{2}}{4} = \frac{-\frac{20}{2}}{4} = -\frac{5}{2}$

Therefore we have come to the solution $x = \begin{pmatrix} -\frac{5}{2} \\ -\frac{5}{16} \\ \frac{17}{8} \end{pmatrix}$

Charthulius Wheezer