Computer Problems

5.1. (*a*) How many zeros does the function

$$f(x) = \sin(10x) - x$$

have? (*Hint*: Sketching the graph of the function will be very helpful.)
(*b*) Use a library routine or one of your own design to find all of the zeros of this function. (*Hint*: You will need a different starting point or initial bracketing interval for each root.)

5.2. For the equation

$$f(x) = x^2 - 3x + 2 = 0,$$

each of the following functions yields an equivalent fixed-point problem:

$$\begin{aligned} g_1(x) &= (x^2 + 2)/3, \\ g_2(x) &= \sqrt{3x - 2}, \\ g_3(x) &= 3 - 2/x, \\ g_4(x) &= (x^2 - 2)/(2x - 3). \end{aligned}$$

(*a*) Analyze the convergence properties of each of the corresponding fixed-point iteration schemes for the root $x = 2$ by considering $|g_i'(2)|$.
(*b*) Confirm your analysis by implementing each of the schemes and verifying its convergence (or lack thereof) and approximate convergence rate.

5.3. Implement the bisection, Newton, and secant methods for solving nonlinear equations in one dimension, and test your implementations by finding at least one root for each of the following equations. What termination criterion should you use? What convergence rate is achieved in each case? Compare your results (solutions and convergence rates) with those for a library routine for solving nonlinear equations.
(*a*) $x^3 - 2x - 5 = 0$.
(*b*) $e^{-x} = x$.
(*c*) $x \sin(x) = 1$.
(*d*) $x^3 - 3x^2 + 3x - 1 = 0$.

5.4. Repeat the previous exercise, this time implementing the inverse quadratic interpolation and linear fractional interpolation methods, and answer the same questions as before.

5.5. Consider the function

$$f(x) = (((x - 0.5) + x) - 0.5) + x,$$

evaluated as indicated (i.e., without any simplification). On your computer, is there any floating-point value x such that $f(x)$ is *exactly* zero? If you use a zero-finding routine on this function, what result is returned, and what is the value of f for this argument? Experiment with the error tolerance to determine its effect on the results obtained.

5.6. Compute the first several iterations of Newton's method for solving each of the following equations, starting with the given initial guess.
(*a*) $x^2 - 1 = 0, \quad x_0 = 10^6$.
(*b*) $(x - 1)^4 = 0, \quad x_0 = 10$.
For each equation, answer the following questions: What is the apparent convergence rate of the sequence initially? What should the asymptotic convergence rate of Newton's method be for this equation? How many iterations are required before the asymptotic range is reached? Give an analytical explanation of the behavior you observe empirically.

5.7. (*a*) How does Newton's method behave when you apply it to find a solution to the nonlinear equation

$$x^5 - x^3 - 4x = 0$$

with $x_0 = 1$ as starting point?
(*b*) What are the real roots of the foregoing equation? Is there anything pathological about them?

5.8. Consider the problem of finding the smallest positive root of the nonlinear equation

$$\cos(x) + 1/(1 + e^{-2x}) = 0.$$

Investigate, both theoretically and empirically, the following iterative schemes for solving this problem using the starting point $x_0 = 3$. For each scheme, you should show that it is indeed an equivalent fixed-point problem, determine analytically whether it is locally convergent and its expected convergence rate, and then implement the method to confirm your results.
(*a*) $x_{k+1} = \arccos(-1/(1 + e^{-2x_k}))$.
(*b*) $x_{k+1} = 0.5 \log(-1/(1 + 1/\cos(x_k)))$.
(*c*) Newton's method.

5.9. In celestial mechanics, *Kepler's equation*

$$M = E - e\sin(E)$$

relates the mean anomaly M to the eccentric anomaly E of an elliptical orbit of eccentricity e, where $0 < e < 1$.

(a) Prove that fixed-point iteration using the iteration function

$$g(E) = M + e\sin(E)$$

is locally convergent.

(b) Use the fixed-point iteration scheme in part a to solve Kepler's equation for the eccentric anomaly E corresponding to a mean anomaly of $M = 1$ (radians) and an eccentricity of $e = 0.5$.

(c) Use Newton's method to solve the same problem.

(d) Use a library zero finder to solve the same problem.

5.10. In neutron transport theory, the critical length of a fuel rod is determined by the roots of the equation

$$\cot(x) = (x^2 - 1)/(2x).$$

Use a zero finder to determine the smallest positive root of this equation.

5.11. The natural frequencies of vibration of a uniform beam of unit length, clamped on one end and free on the other, satisfy the equation

$$\tan(x)\tanh(x) = -1.$$

Use a zero finder to determine the smallest positive root of this equation.

5.12. The vertical distance y that a parachutist falls before opening the parachute is given by the equation

$$y = \log(\cosh(t\sqrt{gk}))/k,$$

where t is the elapsed time in seconds, $g = 9.8065$ m/s^2 is the acceleration due to gravity, and $k = 0.00341$ m^{-1} is a constant related to air resistance. Use a zero finder to determine the elapsed time required to fall a distance of 1 km.

5.13. Use a zero finder to determine the depth (relative to its radius) to which a sphere of density 0.4 (relative to water) sinks. (*Hint*: The volume of a sphere of radius r is $4\pi r^3/3$, and the volume of a spherical "cap" of height h is $\pi h^2(3r - h)/3$).

5.14. The *van der Waals equation of state*,

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT,$$

relates the pressure p, specific volume v, and temperature T of a gas, where R is a universal constant and a and b are constants that depend on the particular gas. In appropriate units, $R = 0.082054$, and for carbon dioxide, $a = 3.592$ and $b = 0.04267$. Use a zero finder to compute the specific volume v for a temperature of 300 K and for pressures of 1 atm, 10 atm, and 100 atm. Compare your results to those for the *ideal gas law*, $pv = RT$. The latter can be used as a starting guess for an iterative method to solve the van der Waals equation.

5.15. If an amount a is borrowed at interest rate r for n years, then the total amount to be repaid is given by

$$a(1+r)^n.$$

Yearly payments of p each would reduce this amount by

$$\sum_{0}^{n-1} p(1+r)^i = p\,\frac{(1+r)^n - 1}{r}.$$

The loan will be repaid when these two quantities are equal.

(a) For a loan of a = \$100,000 and yearly payments of p = \$10,000, how long will it take to pay off the loan if the interest rate is 6 percent, i.e., $r = 0.06$?

(b) For a loan of a = \$100,000 and yearly payments of p = \$10,000, what interest rate r would be required for the loan to be paid off in $n = 20$ years?

(c) For a loan of a = \$100,000, how large must the yearly payments p be for the loan to be paid off in $n = 20$ years at 6 percent interest?

You may use any method you like to solve the given equation in each case. For the purpose of this problem, we will treat n as a continuous variable (i.e., it can have fractional values).