Solution to the 2018 MMF 2021 Exam

1. [10 marks: 5 marks for each part]

I told the students that the function

$$f(x) = \frac{\mathrm{e}^x - 1}{x}$$

satisfies

$$\lim_{x \to 0} f(x) = 1 \tag{1}$$

I also told them that they don't have to prove (1); just accept it as true.

I also gave them a table on page 3 of the exam (see the file exam.2018.pdf) that shows the computed values of $f(x)$ for $x = 10^{-k}$ and $k = 1, 2, \ldots, 15$.

(a) I noted that the computed values for $f(x)$ first seem to be converging to 1 for $k = 1, 2, \ldots, 8$, but then diverge from 1 for $k = 11, 12, \ldots, 15$. I asked them to explain why this happens.

The students should do a little rounding error analysis to explain why the computed values for $f(x)$ in the table behave the way they do. To this end, I told them that they can assume

$$\exp(x) = \mathrm{e}^x(1 + \delta_x)$$

where δ_x changes with x, but its magnitude is at most a few multiples of ϵ_{mach}. (I.e., $|\delta_x| \le c\,\epsilon_{\text{mach}}$ for some c that is at most 2 or 3.)
Therefore,

$$\begin{aligned} \mathrm{fl}(f(x)) &= \mathrm{fl}\left(\frac{\mathrm{e}^x - 1}{x}\right) \\ &= \frac{(\mathrm{e}^x(1 + \delta_x) - 1)(1 + \delta_1)}{x}(1 + \delta_2) \end{aligned} \tag{2}$$

for some δ_1 and δ_2 satisfying $|\delta_1| \le \frac{1}{2}\epsilon_{\text{mach}}$ and $|\delta_2| \le \frac{1}{2}\epsilon_{\text{mach}}$. Now we can perform standard mathematical operations on the last line of (2) to get

$$\begin{aligned} \mathrm{fl}(f(x)) &= \frac{\mathrm{e}^x - 1 + \mathrm{e}^x\delta_x}{x}(1 + \delta_1)(1 + \delta_2) \\ &= \left(\frac{\mathrm{e}^x - 1}{x} + \frac{\delta_x}{x}\mathrm{e}^x\right)(1 + \delta_1)(1 + \delta_2) \\ &= \left(\left(1 + \frac{1}{2}x + \mathcal{O}(x^2)\right) + \left(\frac{\delta_x}{x}\mathrm{e}^x\right)\right)(1 + \delta_1)(1 + \delta_2) \\ &= \left(1 + \left(\frac{1}{2}x + \mathcal{O}(x^2)\right) + \left(\frac{\delta_x}{x}\mathrm{e}^x\right)\right)(1 + \delta_1)(1 + \delta_2) \end{aligned} \tag{3}$$

From our assumption above, $|\delta_x| \leq c\,\epsilon_{\text{mach}} \lessapprox 10^{-15}$. So, for $k = 1, 2, \ldots, 6$ and $x = 10^{-k}$,

$$\left|\frac{\delta_x}{x}\right| \ll \frac{1}{2}x + \mathcal{O}(x^2)$$

Hence, from the last line of (3),

$$\text{fl}(f(x)) \approx 1 + \frac{1}{2}x + \mathcal{O}(x^2)$$

That is, our rounding error analysis predicts that the computed value of $f(x)$ will behave like $1 + \frac{1}{2}x + \mathcal{O}(x^2)$ for $k = 1, 2, \ldots, 6$. We see quite clearly in the table on page 3 of the exam that this is indeed the case.

For the values of k in the range $k = 7, 8, \ldots, 11$, the behaviour of $f(x)$ is not as clear. That's because, for the k in this range,

$$\left|\frac{\delta_x}{x}\right| \approx \frac{1}{2}x + \mathcal{O}(x^2)$$

Hence, from the last line of (3), we see that both

$$\frac{1}{2}x + \mathcal{O}(x^2)$$

and

$$\frac{\delta_x}{x}$$

affect the behaviour of $f(x)$. So, our rounding error analysis predicts that the behaviour of $f(x)$ is not particularly clear in this range. This prediction is supported by the data in the table on page 3 of the exam.

However, for $k = 12, 13, 14, 15$,

$$0 < \frac{1}{2}x + \mathcal{O}(x^2) \ll \left|\frac{\delta_x}{x}\right|$$

So, for this range of k, our rounding error analysis predicts that

$$\text{fl}(f(x)) \approx 1 + \frac{\delta_x}{x}$$

Since the δ_x is somewhat "random" in the range $[-c\,\epsilon_{\text{mach}}, c\,\epsilon_{\text{mach}}]$, the values of $f(x)$ for k in this range are somewhat erratic, but $|\delta_x/x|$ generally grows at x decreases (e.g., k increases). Hence, $f(x)$ diverges from 1 (in a somewhat erratic way) as k increases for k in this range. This prediction is supported by the data in the table on page 3 of the exam.

(b) The students are asked to explain why the computed values for

$$g(x) = \frac{e^x - 1}{\ln(e^x)}$$

shown in column four of the table on page 3 of the exam (see the file exam.2018.pdf) give much more accurate results for small x than $f(x)$ does, even though in exact arithmetic $f(x) = g(x)$ for all $x \in \mathbb{R}$ (assuming you define $f(0) = g(0) = 1$).

To see how rounding errors affect $g(x)$, we first need to see how rounding errors affect $\ln(u)$ for u close to 1. It's reasonable to assume that

$$\mathrm{fl}(\ln(u)) = \ln(u)(1 + \delta_u) \tag{4}$$

However, $|\delta_u|$ might be much larger than ϵ_{mach}, since $\ln(u)$ is ill-conditioned for u close to 1. (Note, we are assuming here that $u = e^x$ and $|x|$ is small, so $u \approx 1$.) For now, let's not try to determine a bound on $|\delta_u|$. We will come back to that later. So, using (4), we can perform a rounding error analysis on $g(x)$ that is much like the one in part (a) for $f(x)$. That is,

$$\begin{aligned} \mathrm{fl}(g(x)) &= \mathrm{fl}\left(\frac{e^x - 1}{\ln(e^x)}\right) \\ &= \frac{(e^x(1 + \delta_x) - 1)(1 + \delta_1)}{\big(\ln(e^x(1 + \delta_x))\big)(1 + \delta_u)}(1 + \delta_2) \end{aligned} \tag{5}$$

for some δ_1 and δ_2 satisfying $|\delta_1| \le \frac{1}{2}\epsilon_{\text{mach}}$ and $|\delta_2| \le \frac{1}{2}\epsilon_{\text{mach}}$. It's important to note that the rounding error that is made when computing e^x is the same for the e^x in the numerator of (5) and the e^x in the denominator of (5). More generally, the rounding error that is made when computing e^x is deterministic. So, the rounding error is the same whenever e^x computed for the same value of x. Therefore, the δ_x in the numerator of (5) is the same as the δ_x in the denominator of (5). This is very important for the analysis below.

For the analysis that follows, it is convenient to note that there is a $\hat{\delta}_x$ such that

$$e^{x + \hat{\delta}_x} = e^x(1 + \delta_x) \tag{6}$$

where by taking logarithms of both sides of (6), we see that

$$x + \hat{\delta}_x = x + \ln(1 + \delta_x)$$

whence

$$\hat{\delta}_x = \ln(1 + \delta_x) = \delta_x + \mathcal{O}(\delta_x^2)$$

Since we assumed in part (a) that $|\delta_x| \le c\,\epsilon_{\text{mach}}$ for some c that is at most 2 or 3, it follows that $|\hat{\delta}_x| \le \hat{c}\,\epsilon_{\text{mach}}$ for some $\hat{c}$ that is only slightly different from c. That is, we can also assume $\hat{c}$ is at most 2 or 3.

Therefore, we can rewrite (5) as

$$
\begin{aligned}
\text{fl}(g(x)) &= \frac{(\mathrm{e}^{x+\hat{\delta}_x} - 1)(1+\delta_1)}{\big(\ln(\mathrm{e}^{x+\hat{\delta}_x})\big)(1+\delta_u)}(1+\delta_2) \\
&= \frac{\mathrm{e}^{x+\hat{\delta}_x} - 1}{\ln(\mathrm{e}^{x+\hat{\delta}_x})} \times \frac{(1+\delta_1)(1+\delta_2)}{(1+\delta_u)} \\
&= \frac{(x+\hat{\delta}_x) + \frac{1}{2}(x+\hat{\delta}_x)^2 + \mathcal{O}((x+\hat{\delta}_x)^3)}{(x+\hat{\delta}_x)} \times \frac{(1+\delta_1)(1+\delta_2)}{(1+\delta_u)} \\
&= \left(1 + \frac{1}{2}(x+\hat{\delta}_x) + \mathcal{O}((x+\hat{\delta}_x)^2\right) \times \frac{(1+\delta_1)(1+\delta_2)}{(1+\delta_u)}
\end{aligned} \tag{7}
$$

For $k = 1, 2, \dots, 13$ and $x = 10^{-k}$,

$$|\hat{\delta}_x| \ll x$$

So,

$$\left(1 + \frac{1}{2}(x+\hat{\delta}_x) + \mathcal{O}((x+\hat{\delta}_x)^2\right) \approx 1 + \frac{1}{2}x \tag{8}$$

which agrees very well with the numerical results shown in the table on page 3 of the exam (see the file exam.2018.pdf). A slightly surprising thing is that the term

$$\frac{(1+\delta_1)(1+\delta_2)}{(1+\delta_u)}$$

on the right in (7) does not disturb the result (8). Although the δ_1 and δ_2 terms would not disturb the result (8), since $|\delta_1| \leq \frac{1}{2}\epsilon_{\text{mach}}$ and $|\delta_2| \leq \frac{1}{2}\epsilon_{\text{mach}}$, I would have expected that the δ_u term could disturb the result (8), since I think we could have $|\delta_u| \gg \epsilon_{\text{mach}}$. However, the results in the table on page 3 of the exam do not suffer from this potentially large perturbation.

Also, for $k = 14, 15$, you might expect that

$$|\hat{\delta}_x| \not\ll x$$

This could also perturb the result (8). However, this potential perturbation does not appear to occur in the numerical results reported in the table on page 3 of the exam.

2. [10 marks: 5 marks for each part]

(a) In this part of the question, the students are told to assume that an integer $n \geq 0$ is given and that a real $\lambda_n \in (0,1)$ is given. They are asked to find a c_{n,λ_n} such that

$$f_n(x)/g_{\lambda_n}(x) \leq c_{n,\lambda_n} \tag{9}$$

for all $x \geq 0$, where

$$f_n(x) = \frac{x^n \mathrm{e}^{-x}}{n!} \qquad \text{for } x \geq 0$$

and

$$g_{\lambda_n}(x) = \lambda_n \mathrm{e}^{-\lambda_n x} \qquad \text{for } x \geq 0$$

As explained in more detail below, we want to find as small a c_{n,λ_n} as possible such that (9) holds. The smallest c_{n,λ_n} can be is

$$c_{n,\lambda_n} = \max_{x \geq 0} \frac{f_n(x)}{g_{\lambda_n}(x)} \tag{10}$$

since, if c_{n,λ_n} were any smaller than the right side of (10), there would be a finite

$$x^*_{n,\lambda_n} = \arg\max_{x \geq 0} \frac{f_n(x)}{g_{\lambda_n}(x)}$$

such that

$$f_n(x^*_{n,\lambda_n})/g_{\lambda_n}(x^*_{n,\lambda_n}) > c_{n,\lambda_n}$$

which violates (9). Hence, the smallest that c_{n,λ_n} can be is given by (10).

To find the maximum of $f_n(x)/g_{\lambda_n}(x)$ for $x \geq 0$, where the integer $n \geq 0$ and the real $\lambda_n \in (0,1)$ are give, let

$$h_{n,\lambda_n}(x) = \frac{f_n(x)}{g_{\lambda_n}(x)} = \frac{x^n \, \mathrm{e}^{-x(1-\lambda_n)}}{\lambda_n \, n!}$$

Consider first the case $n = 0$. In this case,

$$h_{0,\lambda_0}(x) = \frac{\mathrm{e}^{-x(1-\lambda_0)}}{\lambda_0}$$

Since $\lambda_0 \in (0,1)$, $1 - \lambda_0 > 0$. Therefore, $h_{0,\lambda_0}(x)$ is a strictly decreasing function of x for $x \geq 0$. Hence, the maximum of $h_{0,\lambda_0}(x)$, for $x \geq 0$, occurs at $x^*_{0,\lambda_0} = 0$. Thus,

$$c_{0,\lambda_0} = \max_{x \geq 0} \frac{f_0(x)}{g_{\lambda_0}(x)} = \max_{x \geq 0} h_{0,\lambda_0}(x) = h_{0,\lambda_0}(0) = 1/\lambda_0$$

That is,

$$c_{0,\lambda_0} = 1/\lambda_0 \tag{11}$$

Now consider the case $n \geq 1$. In this case,

$$h_{n,\lambda_n}(x) = \frac{x^n \, \mathrm{e}^{-x(1-\lambda_n)}}{\lambda_n \, n!}$$

satisfies $h_{n,\lambda_n}(0) = 0$,

$$\lim_{x \to \infty} h_{n,\lambda_n}(x) = 0 \qquad (\text{since } 1 - \lambda_n > 0)$$

and $h_{n,\lambda_n}(x) > 0$ for $x \in (0, \infty)$. So, $h_{n,\lambda_n}(x)$ must have a maximum at some $x^*_{n,\lambda_n} \in (0, \infty)$. To find x^*_{n,λ_n}, differentiate $h_{n,\lambda_n}(x)$ with respect to x and solve $h'_{n,\lambda_n}(x^*_{n,\lambda_n}) = 0$. To this end, note that

$$\begin{aligned} h'_{n,\lambda_n}(x) &= \frac{n x^{n-1} \mathrm{e}^{-x(1-\lambda_n)} - x^n (1-\lambda_n) \mathrm{e}^{-x(1-\lambda_n)}}{\lambda_n \, n!} \\ &= \frac{x^{n-1} \mathrm{e}^{-x(1-\lambda_n)} \big(n - x(1-\lambda_n)\big)}{\lambda_n \, n!} \end{aligned}$$

Now note that $h'_{n,\lambda_n}(x^*_{n,\lambda_n}) = 0$ and $x^*_{n,\lambda_n} \in (0, \infty)$ has only one solution

$$x^*_{n,\lambda_n} = \frac{n}{1 - \lambda_n}$$

Therefore, for $n \geq 1$,

$$c_{n,\lambda_n} = \max_{x \geq 0} \frac{f_n(x)}{g_{\lambda_n}(x)} = \max_{x \geq 0} h_{n,\lambda_n}(x) = h_{n,\lambda_n}(x^*_{n,\lambda_n}) = \left(\frac{n}{1 - \lambda_n} \right)^n \frac{\mathrm{e}^{-n}}{\lambda_n \, n!}$$

That is,

$$c_{n,\lambda_n} = \left(\frac{n}{1 - \lambda_n} \right)^n \frac{\mathrm{e}^{-n}}{\lambda_n \, n!} \tag{12}$$

The students are asked to explain why they think the value for c_{n,λ_n} they chose is the best choice for the given $n \geq 0$ and the given $\lambda_n \in (0, 1)$.

As noted above, the c_{n,λ_n} must satisfy (9). However, you want to choose as small a c_{n,λ_n} as possible, since the probability of an acceptance in the acceptance-rejection method is $1/c_{n,\lambda_n}$. So, if you choose c_{n,λ_n} as small as possible, you make the acceptance rate as large as possible and, the larger the acceptance rate, the more efficient the acceptance-rejection method is.

As explained above, the smallest that c_{n,λ_n} can be, while satisfying (9), is given by (10).

Given the c_{n,λ_n} determined above, the acceptance-rejection method to compute a random variable X having the pdf $f_n(x)$ is as follows.

1. $U = \text{rand}(1,2)$ (Generate $U(1)$ and $U(2)$ independent Unif[0,1] random variables.)
2. Set $Y = \frac{1}{\lambda_n} \ln(U(1))$ (Y is an exponential random variable with pdf $g_{\lambda_n}(x)$.)

   (In MatLab, replace ln by log.)
3. If $U(2) \leq \dfrac{f_n(Y)}{c_{n,\lambda_n}\, g_{\lambda_n}(Y)}$

   then return $X = Y$ (Accept X.)

   else go to step 1 (Reject)

The students don't have to include the comments that I have included above. Accept anything that seems reasonable to you.

(b) For $n = 0$, we have from (11)

$$c_{0,\lambda_0} = 1/\lambda_0$$

and we also have the constraint $\lambda_0 \in (0, 1)$. Since we want c_{0,λ_0} to be as small as possible, we want to choose λ_0 to be as large as possible. However, we must choose $\lambda_0 < 1$. So, we should pick a λ_0 close to 1, such as $\lambda_0 = 0.9999$. With this choice of λ_0,

$$c_{0,\lambda_0} = 1/\lambda_0 \approx 1.0001$$

Note that the acceptance rate in this case is

$$\frac{1}{c_{0,\lambda_0}} = \lambda_0 = 0.9999$$

Hence, you almost always accept. So, this acceptance-rejection method is very efficient.

[Aside: you don't really need to use the acceptance-rejection method for $n = 0$, since in this case $f_0(x) = \mathrm{e}^{-x}$. So, $f(x)$ is the pdf of an exponential random variable. So, you can just compute $X = -\ln(U)$. Picking $\lambda_0 = 0.9999$ makes $g_{\lambda_0}(x) \approx f_0(x)$ for all $x \geq 0$. So, the acceptance-rejection method works very well in this case, even though it would be better to generate X as $X = -\ln(U)$.]

For $n \geq 1$, we have from (12)

$$c_{n,\lambda_n} = \left(\frac{n}{1-\lambda_n}\right)^n \frac{\mathrm{e}^{-n}}{\lambda_n\, n!} \tag{13}$$

For a given $n \geq 1$, we want to choose $\lambda_n \in (0,1)$ to make c_{n,λ_n} given by (13) as small as possible. (As explained above, this makes the acceptance-rejection method as efficient as possible.) To this end, let

$$c(\lambda) = \left(\frac{n}{1-\lambda}\right)^n \frac{\mathrm{e}^{-n}}{\lambda\, n!}$$

and attempt to find the minimum of $c(\lambda)$ for $\lambda \in (0,1)$.

Note $c(\lambda) \to \infty$ as $\lambda \to 0$ and also as $\lambda \to 1$ and $c(\lambda)$ is finite and positive for all $\lambda \in (0,1)$. Therefore $c(\lambda)$ has a minimum at some $\lambda_n^* \in (0,1)$.

To find the minimum of $c(\lambda)$ for $\lambda \in (0,1)$, differentiate $c(\lambda)$ with respect to λ and solve $c'(\lambda_n^*) = 0$ for λ_n^*. To this end, note that

$$\begin{aligned} c'(\lambda) &= n \left(\frac{n}{1-\lambda}\right)^{n-1} \frac{n}{(1-\lambda)^2} \frac{\mathrm{e}^{-n}}{\lambda\, n!} - \left(\frac{n}{1-\lambda}\right)^n \frac{\mathrm{e}^{-n}}{\lambda^2\, n!} \\ &= \left(\frac{n}{1-\lambda}\right)^n \frac{\mathrm{e}^{-n}}{\lambda\, n!} \left(\frac{n}{1-\lambda} - \frac{1}{\lambda}\right) \end{aligned}$$

The only solution of $c'(\lambda_n^*) = 0$ is

$$\lambda_n^* = \frac{1}{n+1}$$

For this λ_n^*,

$$c_{n,\lambda_n^*} = c(\lambda_n^*) = \frac{\mathrm{e}^{-n}\,(n+1)^{n+1}}{n!}$$

Note that this is the value α_n given on page 5 of the exam. So, $c_{n,\lambda_n^*} = \alpha_n$ is the optimal choice of c_{n,λ_n} for $n \geq 1$.

If n is fairly small (e.g., $1 \leq n \leq 5$), then the table for α_n values on page 5 of the exam shows that $1.4715 \leq c_{n,\lambda_n^*} = \alpha_n \leq 2.6197$. Hence, the acceptance rate (i.e., $1/c_{n,\lambda_n^*}$) varies from about 0.67 (for $n = 1$) to about 0.38 (for $n = 5$). So, the acceptance-rejection method is reasonably efficient for n in the range 1 to 5.

However, the acceptance rate for the acceptance-rejection method decreases as n increases. For n in the range 6 to 20, $c_{n,\lambda_n^*} = \alpha_n$ increases from about 2.8352 to about 4.9498. Hence, the acceptance rate (i.e., $1/c_{n,\lambda_n^*}$) varies from about 0.35 (for $n = 6$) to about 0.20 (for $n = 20$). This is still tolerable, but not particularly good.

However, for large n, I told them that $c_{n,\lambda_n^*} = \alpha_n \approx 1.1\sqrt{n}$. Hence, the acceptance rate (i.e., $1/c_{n,\lambda_n^*}$) is about $0.9/\sqrt{n}$. Although this is not too bad for $n = 21$, for $n = 100$, the acceptance rate is about 0.09, which is starting to get too small. As n continues to grow, the acceptance rate continues to get smaller. So, the method becomes very inefficient for large n. For example, for $n = 1{,}000{,}000$, the acceptance rate is about 0.0009, which makes the method very inefficient.

3. [10 marks]

As noted in the question, the option price at time $t = 0$ is

$$P_0 = \mathbb{E}[\mathrm{e}^{-rT} h(S_T^{(1)}, S_T^{(2)})] \tag{14}$$

where

$$h(S_T^{(1)}, S_T^{(2)}) = \max(\omega_1 S_T^{(1)} + \omega_2 S_T^{(2)} - \hat{K}, 0) \tag{15}$$

where ω_1 and ω_2 are real constants satisfying $\omega_1 \in [0,1]$, $\omega_2 \in [0,1]$, $\omega_1 + \omega_2 = 1$ and $\hat{K} \in \mathbb{R}$ is a positive constant. So, combining (14) and (15), we get

$$P_0 = \mathbb{E}[\mathrm{e}^{-rT} \max(\omega_1 S_T^{(1)} + \omega_2 S_T^{(2)} - \hat{K}, 0)] \tag{16}$$

Also, the question notes that

$$\begin{aligned} S_T^{(1)} &= S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2/2)T + \sigma_1 W_T^{(1)}} \\ S_T^{(2)} &= S_0^{(2)} \mathrm{e}^{(r-\sigma_2^2/2)T + \sigma_2 W_T^{(2)}} \end{aligned}$$

and the correlated Brownian motions $W_T^{(1)}$ and $W_T^{(2)}$ satisfy

$$\begin{aligned} W_T^{(1)} &= \sqrt{T}\left(\sqrt{1-\rho^2} Z^{(1)} + \rho Z^{(2)}\right) \\ W_T^{(2)} &= \sqrt{T} Z^{(2)} \end{aligned}$$

where $Z^{(1)} \sim N(0,1)$, $Z^{(2)} \sim N(0,1)$ and $Z^{(1)}$ and $Z^{(2)}$ are independent. Therefore,

$$\begin{aligned} S_T^{(1)} &= S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2/2)T + \sigma_1 \sqrt{T}\left(\sqrt{1-\rho^2} Z^{(1)} + \rho Z^{(2)}\right)} \\ S_T^{(2)} &= S_0^{(2)} \mathrm{e}^{(r-\sigma_2^2/2)T + \sigma_2 \sqrt{T} Z^{(2)}} \end{aligned} \tag{17}$$

where $Z^{(1)} \sim N(0,1)$, $Z^{(2)} \sim N(0,1)$ and $Z^{(1)}$ and $Z^{(2)}$ are independent. Substituting the $S_T^{(1)}$ and $S_T^{(2)}$ from (17) into (16), we get

$$\begin{aligned} P_0 = \mathbb{E}[\mathrm{e}^{-rT} \max(&\omega_1 S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2/2)T + \sigma_1 \sqrt{T}\left(\sqrt{1-\rho^2} Z^{(1)} + \rho Z^{(2)}\right)} \\ &+ \omega_2 S_0^{(2)} \mathrm{e}^{(r-\sigma_2^2/2)T + \sigma_2 \sqrt{T} Z^{(2)}} - \hat{K}, 0)] \end{aligned} \tag{18}$$

Now we can apply conditional expectation to (18) to get

$$\begin{aligned} P_0 = \mathbb{E}_{Z^{(2)}}[\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT} \max(&\omega_1 S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2/2)T + \sigma_1 \sqrt{T}\left(\sqrt{1-\rho^2} Z^{(1)} + \rho Z^{(2)}\right)} \\ &+ \omega_2 S_0^{(2)} \mathrm{e}^{(r-\sigma_2^2/2)T + \sigma_2 \sqrt{T} Z^{(2)}} - \hat{K}, 0) | Z^{(2)}]] \end{aligned} \tag{19}$$

where the outer expectation is with respect to $Z^{(2)}$ and the inner expectation is with respect to $Z^{(1)}$. To clarify (19) a little, we introduce a deterministic variable z_2 and re-write (19) as

$$\begin{aligned} P_0 = \mathbb{E}_{Z^{(2)}}[\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT} \max(&\omega_1 S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2/2)T + \sigma_1 \sqrt{T}\left(\sqrt{1-\rho^2} Z^{(1)} + \rho z_2\right)} \\ &+ \omega_2 S_0^{(2)} \mathrm{e}^{(r-\sigma_2^2/2)T + \sigma_2 \sqrt{T} z_2} - \hat{K}, 0) | z_2 = Z^{(2)}]] \end{aligned} \tag{20}$$

Now note that

$$
\begin{aligned}
&\omega_1 S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}\left(\sqrt{1-\rho^2}Z^{(1)}+\rho z_2\right)} \\
&\quad = \omega_1 S_0^{(1)} \mathrm{e}^{(r-\sigma_1^2(1-\rho^2)/2)T-\sigma_1^2\rho^2 T/2+\sigma_1\sqrt{1-\rho^2}\sqrt{T}Z^{(1)}+\sigma_1\rho\sqrt{T}z_2} \\
&\quad = \omega_1 S_0^{(1)} \mathrm{e}^{-\sigma_1^2\rho^2 T/2+\sigma_1\rho\sqrt{T}z_2} \mathrm{e}^{(r-\sigma_1^2(1-\rho^2)/2)T+\sigma_1\sqrt{1-\rho^2}\sqrt{T}Z^{(1)}} \\
&\quad = S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}}
\end{aligned}
$$

where

$$
\begin{aligned}
S_0(z_2) &= \omega_1 S_0^{(1)} \mathrm{e}^{-\sigma_1^2\rho^2 T/2+\sigma_1\rho\sqrt{T}z_2} \\
\sigma &= \sigma_1\sqrt{1-\rho^2}
\end{aligned}
\tag{21}
$$

In addition, let

$$
K(z_2) = \hat{K} - \sigma_2 S_0^{(2)} \mathrm{e}^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}z_2} \tag{22}
$$

So, we can re-write the inner expectation in (20) as

$$
\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT}\max(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2), 0)] \tag{23}
$$

There is a complication here that I didn't notice when I wrote the question: we can use the Black-Scholes formula to evaluate (23) if $K(z_2) > 0$, but I don't think the Black-Scholes formula is applicable if $K(z_2) < 0$. Take off one mark only if the students fail to notice this and use the the Black-Scholes formula to evaluate (23) without the modification I describe below.

To handle the complication noted above for $K(z_2) < 0$, notice that, if $K(z_2) \le 0$, then

$$
S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2) \ge 0
$$

Hence, if $K(z_2) \le 0$, then

$$
\max(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2), 0) = S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2)
$$

Therefore, we can rewrite (23) as

$$
\begin{aligned}
\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT}\Big(&\max(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2), 0)\mathbb{1}_{K(z_2)>0} \\
&+ \left(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2)\right)\mathbb{1}_{K(z_2)\le 0}\Big)]
\end{aligned}
\tag{24}
$$

We can rewrite (24) in turn as

$$
\begin{aligned}
&\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT}\max(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2), 0)]\mathbb{1}_{K(z_2)>0} \\
&+ \mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT}\left(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2)\right)]\mathbb{1}_{K(z_2)\le 0}
\end{aligned}
\tag{25}
$$

The first expectation in (25) is the price of a “vanilla” call option with strike price $K(z_2) > 0$, expiry $t = T$ and underlying asset S_t that starts from $S_0(z_2)$ at time $t = 0$ and evolves in time according to the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Therefore, if $K(z_2) > 0$,

$$\text{Call} = \mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT} \max(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2), 0)]$$

where

$$\texttt{[Call, Put] = blsprice(}S_0(z_2)\texttt{, }K(z_2)\texttt{, }r\texttt{, }T\texttt{, }\sigma\texttt{)}$$

and $\sigma = \sigma_1\sqrt{1-\rho^2}$. So, write a little wrapper function `blspriceCall` such that

$$\texttt{Call = blspriceCall(}S_0(z_2)\texttt{, }K(z_2)\texttt{, }r\texttt{, }T\texttt{, }\sigma\texttt{)}$$

where `Call` is given above by `blsprice` with the same parameters.

On the other hand, a simple integration with the normal distribution shows that the second expectation in (25) can be evaluated as

$$\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT}\big(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2)\big)] = S_0(z_2) - \mathrm{e}^{-rT}K(z_2)$$

Therefore, (23) can be evaluated as follows:

$$\begin{aligned}&\mathbb{E}_{Z^{(1)}}[\mathrm{e}^{-rT} \max(S_0(z_2)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z^{(1)}} - K(z_2), 0)]\\ &= \texttt{blspriceCall}\big(S_0(z_2)\texttt{, }K(z_2)\texttt{, }r\texttt{, }T\texttt{, }\sigma\big)\mathbb{1}_{K(z_2)>0}\\ &+ \big(S_0(z_2) - \mathrm{e}^{-rT}K(z_2)\big)\mathbb{1}_{K(z_2)\le 0}\end{aligned} \tag{26}$$

So, we can use (26) to rewrite (20) as

$$\begin{aligned}P_0 = \mathbb{E}_{Z^{(2)}}[&\texttt{blspriceCall}\big(S_0(Z^{(2)})\texttt{, }K(Z^{(2)})\texttt{, }r\texttt{, }T\texttt{, }\sigma\big)\mathbb{1}_{K(Z^{(2)})>0}\\ &+ \big(S_0(Z^{(2)}) - \mathrm{e}^{-rT}K(Z^{(2)})\big)\mathbb{1}_{K(Z^{(2)})\le 0}]\end{aligned} \tag{27}$$

We can use (27) as the basis for the Monte Carlo simulation below to price this *basket option.*

for $n = 1, 2, \ldots, N$,
    Generate $Z_n \sim N(0, 1)$ using a function such as MatLab's `randn`
    if $K(Z_n) > 0$ then
        $Y_n =$ `blspriceCall(`$S_0(Z_n)$`,` $K(Z_n)$`,` r`,` T`,` σ`)`
    else
        $Y_n = S_0(Z_n) - \mathrm{e}^{-rT} K(Z_n)$
    end if
end for

Approximate the option price P_0 by
    $\hat{P}_0 = \frac{1}{N} \sum_{n=1}^{N} Y_n$

Note that, in the pseudo-code above, $S_0(z)$ and σ are given in (21) and $K(z)$ is given in (22).

4. [13 marks: 3 marks for part (a) and 5 marks for each of parts (b) and (c)]

This question focuses on the mixed PDE

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2} \tag{28}$$

for $u(t, x, y)$, where $t \in (0, T)$, $x \in (0, 1)$ and $y \in (0, 1)$. Assume that you are given an initial condition at $t = 0$

$$u(0, x, y) = u_0(x, y) \quad \text{for } x \in [0, 1] \text{ and } y \in [0, 1]$$

and appropriate boundary conditions for $t \in (0, T]$.

(a) The students are asked to give a consistent and stable numerical method for (28). I probably should have told them to give an explicit method. The method I give below is explicit. However, since I didn't stipulated an explicit method, they could give an implicit method, such as the fully implicit method or the Crank-Nicholson method.

The only method that we discussed in class for discretizing the $\partial^2 u/\partial y^2$ term is the central difference formula for second-order derivatives. So, I expect that they will likely use the central difference formula for this term, as I do below.

We discussed in class three methods for discretizing the $\partial u/\partial x$ term. I use the backward difference formula for this term, since I think it is the only formula of the three that is stable for an explicit method.

It will probably be very easy to see if their method is consistent, but it might be trickier to see if it is stable. Possibly the easiest way to check if it is stable is to just program the method and test it. It should be very easy to see if it is not stable. Also, there should not be many different numerical methods to consider. If you have any trouble with this, let me know and I will help.

Take off two marks if the method is not consistent and take off two marks if the method is not stable. However, if the method is neither consistent nor stable, give them 0, not -1.

Since $u(t, x, y)$ has three variables, let $u_{i,j}^k \approx u(k\Delta t, i\Delta x, j\Delta y)$. Given this notation and the discussion above, my explicit numerical method is

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} + \frac{u_{i,j}^k - u_{i-1,j}^k}{\Delta x} = \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{(\Delta y)^2} \tag{29}$$

They could also write the method in the equivalent form

$$u_{i,j}^{k+1} = u_{i,j}^k - \frac{\Delta t}{\Delta x}\left(u_{i,j}^k - u_{i-1,j}^k\right) + \frac{\Delta t}{(\Delta y)^2}\left(u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k\right) \tag{30}$$

(b) To determine the order of consistency of a numerical method for a PDE, such as (28), it is much better to work with the numerical method in the form (29), rather than (30). If you work with (30), it is easy to get the order of consistency in t one too high.

So, let's work with the numerical method in the form (29). To determine the order of consistency, move all the terms to the left side and substitute in the exact solution for all the approximate values $u_{i,j}^k$. To make the notation a little easier, let $u_{i,j}^k \approx u(k\Delta t, i\Delta x, j\Delta y) = u(t,x,y)$. That is, to simplify the notation below, use $t = k\Delta t$, $x = i\Delta x$ and $y = j\Delta y$. Following this approach, we get

$$\begin{aligned} &\frac{u(t+\Delta t,x,y) - u(t,x,y)}{\Delta t} \\ &\qquad + \frac{u(t,x,y) - u(t,x-\Delta x,y)}{\Delta x} \\ &\qquad - \frac{u(t,x,y+\Delta y) - 2u(t,x,y) + u(t,x,y-\Delta y)}{(\Delta y)^2} \\ &\quad = u_t(t,x,y) + \mathcal{O}(\Delta t) + u_x(t,x,y) + \mathcal{O}(\Delta x) - u_{yy}(t,x,y) + \mathcal{O}((\Delta y)^2) \end{aligned} \tag{31}$$

They don't have to actually show that

$$\frac{u(t+\Delta t,x,y) - u(t,x,y)}{\Delta t} = u_t(t,x,y) + \mathcal{O}(\Delta t)$$

for example. We did this in class. So, just accept this if they write it without proof. Similarly, just accept the other derivative approximations if they are correct.

Now note that, since $u(t,x,y)$ is the solution of the PDE (28),

$$u_t(t,x,y) + u_x(t,x,y) - u_{yy}(t,x,y) = 0$$

Hence, (31) reduces to

$$\begin{aligned} &\frac{u(t+\Delta t,x,y) - u(t,x,y)}{\Delta t} \\ &\qquad + \frac{u(t,x,y) - u(t,x-\Delta x,y)}{\Delta x} \\ &\qquad - \frac{u(t,x,y+\Delta y) - 2u(t,x,y) + u(t,x,y-\Delta y)}{(\Delta y)^2} \\ &\quad = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x) + \mathcal{O}((\Delta y)^2) \end{aligned} \tag{32}$$

Therefore, the numerical method (29) is first-order consistent in t, first-order consistent in x and second-order consistent in y.

(c) We want to show that our numerical method (29), or equivalently (30), is stable. There are several ways that the students could do this.

One thing you have to be careful about, though, is that the eigenvalue method I showed them in class may not work in this case to show that (30) is stable, because the associated iteration matrix may not have a full set of eigenvectors. This is the case if they use either a forward or backward difference to discretize the $\partial u/\partial x$ term. That is, for these differences, the iteration matrix is *defective*. The eigenvalue method that we used in class requires that we write out the error vector in terms of the eigenvectors of the iteration matrix. This is not possible if the iteration matrix is defective.

(The eigenvalue method may work if they use a central difference for the $\partial u/\partial x$ term, but I'm not completely sure about this.)

I think the easiest way to show that the method (29) is stable is to show that the iteration matrix associated with the form (30) has norm 1. I showed the students in class that in this case the method is stable. (See my lecture notes.)

For this approach, for each time level k, you put all the values $u_{i,j}^k$ into a vector and then you can write (30) in vector form as

$$u^{k+1} = Au^k + b^k$$

where b^k contains the boundary values. Now consider a "generic" row of A. It will have the terms

$$\frac{\Delta t}{(\Delta y)^2} \quad \cdots \quad \frac{\Delta t}{\Delta x} \quad \left(1 - \frac{\Delta t}{\Delta x} - 2\frac{\Delta t}{(\Delta y)^2}\right) \quad \cdots \quad \frac{\Delta t}{(\Delta y)^2}$$

where the term

$$\left(1 - \frac{\Delta t}{\Delta x} - 2\frac{\Delta t}{(\Delta y)^2}\right)$$

is on the diagonal and the other terms are in different off-diagonal elements. Some rows that correspond to u_{ij}^k terms that are beside the boundary may be missing some of the off-diagonal terms.

Now consider the infinity-norm (i.e., the max-norm) of A. If you take the absolute value of all the elements in the generic row above and sum them across the row, you get

$$\left|1 - \frac{\Delta t}{\Delta x} - 2\frac{\Delta t}{(\Delta y)^2}\right| + 2\frac{\Delta t}{(\Delta y)^2} + \frac{\Delta t}{\Delta x} \tag{33}$$

If

$$1 - \frac{\Delta t}{\Delta x} - 2\frac{\Delta t}{(\Delta y)^2} \geq 0 \tag{34}$$

then (33) reduces to

$$1 - \frac{\Delta t}{\Delta x} - 2\frac{\Delta t}{(\Delta y)^2} + 2\frac{\Delta t}{(\Delta y)^2} + \frac{\Delta t}{\Delta x} = 1 \tag{35}$$

For the rows that correspond to u_{ij}^k terms that are beside the boundary the corresponding sum will be < 1.

Hence, from (35), $\|A\|_\infty = 1$, if constraint (34) is satisfied. Note also that the constraint (34) is equivalent to

$$\frac{\Delta t}{\Delta x} + 2\frac{\Delta t}{(\Delta y)^2} \leq 1 \tag{36}$$

which is similar to the constraint

$$\frac{\Delta t}{(\Delta y)^2} \leq \frac{1}{2}$$

that we got for the heat equation. Therefore, the numerical method (29), or equivalently (30), is stable if (34), or equivalently (36), is satisfied.

I think the numerical method (29), or equivalently (30), is not stable if (34), or equivalently (36), is not satisfied, but it is not necessary for the students to show this. For an explicit method, they just need to get some condition on the Δt, Δx and Δy that ensures that the method is stable.

If their method is implicit, they may not need a constraint on the Δy, but they may still need a constraint on the Δt and Δx.

If you have trouble marking any of the solutions that the students give you, let me know and I will help.