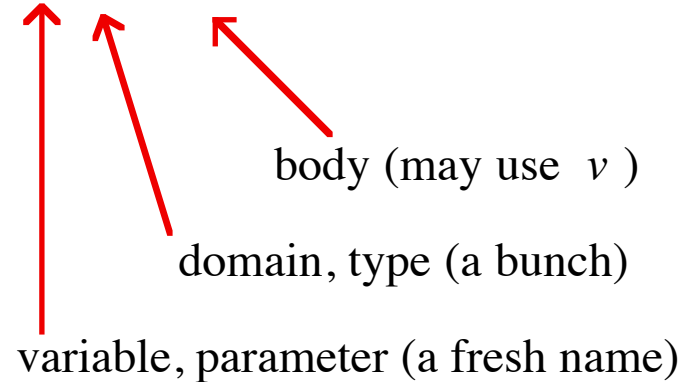


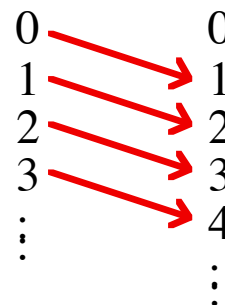
Function Theory

$\langle v: D \rightarrow b \rangle$ “map v in D to b ”



$v: D$ is a local axiom within b

$\langle n: nat \rightarrow n+1 \rangle$



Renaming

$$\langle n: \text{nat} \rightarrow n+1 \rangle = \langle m: \text{nat} \rightarrow m+1 \rangle$$

Domain

$$\Delta f \quad \text{“domain of } f \text{”}$$

$$\Delta \langle n: \text{nat} \rightarrow n+1 \rangle = \text{nat}$$

Application

$$f x \quad \text{“} f \text{ applied to } x \text{” or “} f \text{ of } x \text{”}$$

$$f(x) \quad (f)x \quad f(x+1) \quad -x \quad \neg x$$

$$\langle n: \text{nat} \rightarrow n+1 \rangle 3 = 3+1 = 4$$

two variables

$$\mathit{max} = \langle x: \mathit{xrat} \rightarrow \langle y: \mathit{xrat} \rightarrow \mathbf{if} \ x \geq y \ \mathbf{then} \ x \ \mathbf{else} \ y \rangle \rangle$$

$$\mathit{max} \ 3 = \langle y: \mathit{xrat} \rightarrow \mathbf{if} \ 3 \geq y \ \mathbf{then} \ 3 \ \mathbf{else} \ y \rangle$$

$$\mathit{max} \ 3 \ 5 = (\mathbf{if} \ 3 \geq 5 \ \mathbf{then} \ 3 \ \mathbf{else} \ 5) = 5$$

$$\mathit{max}(3, 5) = \mathit{max} \ 3, \mathit{max} \ 5$$

predicate function with boolean result

$$\mathit{even} = \langle i: \mathit{int} \rightarrow i/2: \mathit{int} \rangle$$

relation function with predicate result

$$\mathit{divides} = \langle n: \mathit{nat}+1 \rightarrow \langle i: \mathit{int} \rightarrow i/n: \mathit{int} \rangle \rangle$$

$$\mathit{even} = \mathit{divides} \ 2$$

selective union

$f | g$ “ f otherwise g ”

$$\Delta(f | g) = \Delta f, \Delta g$$

$$(f | g) x = \mathbf{if } x: \Delta f \mathbf{ then } f x \mathbf{ else } g x$$

abbreviated function notations

$$\langle x: xrat \rightarrow \langle y: xrat \rightarrow \mathbf{if } x \geq y \mathbf{ then } x \mathbf{ else } y \rangle \rangle = \langle x, y: xrat \rightarrow \mathbf{if } x \geq y \mathbf{ then } x \mathbf{ else } y \rangle$$

$$\langle n: nat \rightarrow n+1 \rangle = \langle n \rightarrow n+1 \rangle$$

$$\langle n: 2 \rightarrow 3 \rangle = 2 \rightarrow 3 \quad \text{scope brackets go with variable}$$

$$\langle x: int \rightarrow \langle y: int \rightarrow x+3 \rangle \rangle = x+3 \quad ? \quad \text{but we can't apply it}$$

Scope and Substitution

local

bound, hidden, private

introduction is inside the expression (formal)

nonlocal

global, free, visible, public

introduction is outside the expression (formal or informal)

$$\langle x \rightarrow x \ y \ \rangle (\ x \ y \)$$

Scope and Substitution

local

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introduction is outside the expression (formal or informal)

$$\begin{aligned} & \langle x \rightarrow x \rangle \langle x \rightarrow x \rangle x \rangle 3 \\ = & (3 \langle x \rightarrow x \rangle 3) \end{aligned}$$

Scope and Substitution

local

bound, hidden, private

introduction is inside the expression (formal)

nonlocal

global, free, visible, public

introduction is outside the expression (formal or informal)

$$\begin{aligned} & \langle y \rightarrow x \ y \ \langle x \rightarrow x \ y \ \rangle \ x \ y \ \rangle x && \text{rename inner } x \text{ to } z \\ = & \langle y \rightarrow x \ y \ \langle z \rightarrow z \ y \ \rangle \ x \ y \ \rangle x && \text{now apply} \\ = & (\ x \ x \ \langle z \rightarrow z \ x \ \rangle \ x \ x \) \end{aligned}$$

Quantifiers

A quantifier is an operator that applies to a function.

It is defined from a two-operand symmetric associative operator.

$\forall p$ is defined from \wedge

$\forall \langle r: \text{rat} \rightarrow r < 0 \vee r = 0 \vee r > 0 \rangle$

$\exists p$ is defined from \vee

$\exists \langle n: \text{nat} \rightarrow n = 0 \rangle$

Σf is defined from $+$

$\Sigma \langle n: \text{nat}+1 \rightarrow 1/2^n \rangle$

Πf is defined from \times

$\Pi \langle n: \text{nat}+1 \rightarrow (4 \times n^2) / (4 \times n^2 - 1) \rangle$

abbreviations

$\forall r: \text{rat} \cdot r < 0 \vee r = 0 \vee r > 0$

$\Sigma n: \text{nat}+1 \cdot 1/2^n$

$\forall x, y: \text{rat} \cdot x = y+1 \Rightarrow x > y = \forall x: \text{rat} \cdot \forall y: \text{rat} \cdot x = y+1 \Rightarrow x > y$

$\Sigma n, m: 0, \dots, 10 \cdot n \times m = \Sigma n: 0, \dots, 10 \cdot \Sigma m: 0, \dots, 10 \cdot n \times m$

$$\forall v: \text{null} \cdot b = \top$$

$$\forall v: x \cdot b = \langle v: x \rightarrow b \rangle x$$

$$\forall v: A, B \cdot b = (\forall v: A \cdot b) \wedge (\forall v: B \cdot b)$$

$$\exists v: \text{null} \cdot b = \perp$$

$$\exists v: x \cdot b = \langle v: x \rightarrow b \rangle x$$

$$\exists v: A, B \cdot b = (\exists v: A \cdot b) \vee (\exists v: B \cdot b)$$

$$\Sigma v: \text{null} \cdot n = 0$$

$$\Sigma v: x \cdot n = \langle v: x \rightarrow n \rangle x$$

$$(\Sigma v: A, B \cdot n) + (\Sigma v: A' B \cdot n) = (\Sigma v: A \cdot n) + (\Sigma v: B \cdot n)$$

$$\Pi v: \text{null} \cdot n = 1$$

$$\Pi v: x \cdot n = \langle v: x \rightarrow n \rangle x$$

$$(\Pi v: A, B \cdot n) \times (\Pi v: A' B \cdot n) = (\Pi v: A \cdot n) \times (\Pi v: B \cdot n)$$

build your own

$$\text{MAX } x: \text{rat} \cdot 4 \times x - x^2 = 4$$

$$\text{MAX } v: \text{null} \cdot n = -\infty$$

$$\text{MAX } v: x \cdot n = \langle v: x \rightarrow n \rangle x$$

$$\text{MAX } v: A, B \cdot n = \max (\text{MAX } v: A \cdot n) (\text{MAX } v: B \cdot n)$$

$$x \wedge \top = x$$

$$x \vee \perp = x$$

$$x + 0 = x$$

$$x \times 1 = x$$

$$\max x (-\infty) = x$$

Solution Quantifier

$\S p$ is the (bunch of) solutions of predicate p

$$\S v: \text{null} \cdot b = \text{null}$$

$$\S v: x \cdot b = \mathbf{if} \langle v: x \rightarrow b \rangle x \mathbf{then} x \mathbf{else} \text{null}$$

$$\S v: A, B \cdot b = (\S v: A \cdot b), (\S v: B \cdot b)$$

$$\{ \S i: \text{int} \cdot i^2=4 \} = \{-2, 2\}$$

$$\{ \S n: \text{nat} \cdot n < 3 \} = \{0, \dots, 3\}$$

An expression talks about its nonlocal variables.

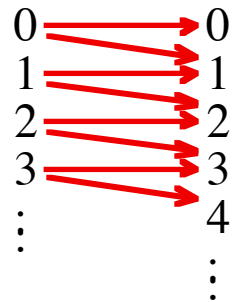
$\exists n: nat \cdot x = 2 \times n$

says

“ x is an even natural”

Function Fine Points

partial:	sometimes no result
total:	always at least one result
deterministic:	always at most one result
nondeterministic:	sometimes more than one result
$\langle n: \text{nat} \cdot n, n+1 \rangle$	total and nondeterministic



$$\langle n: \text{nat} \cdot n, n+1 \rangle 3 = 3, 4$$

distribution

$$(f, g) x = fx, gx$$

$$f(x, y) = fx, fy$$

$$\text{double} = \langle n: \text{nat} \rightarrow n+n \rangle$$

$$\text{double } 2 = 4$$

$$\text{double } (2, 3) = \text{double } 2, \text{double } 3 = 4, 6$$

$$\text{double } (2, 3) \neq (2, 3) + (2, 3) = 4, 5, 6$$

$$\text{tiny} = \langle S: \text{!nat} \rightarrow \$S < 3 \rangle$$

$$\text{tiny } \{\text{null}\} = \top$$

$$\text{tiny } \{0, 1, 2, 3\} = \perp$$

$$\text{tiny } \text{null} = \text{null}$$

function inclusion

$$f: g = \Delta g: \Delta f \wedge \forall x: \Delta g \cdot fx: gx$$

$$A \rightarrow B = \langle a: A \rightarrow B \rangle$$

$A \rightarrow B$ is a function whose domain is A and whose result is B .

$$f: A \rightarrow B = A: \Delta f \wedge \forall a: A \cdot fa: B$$

$A \rightarrow B$ is all those functions whose domain is at least A and whose result is at most B .

$A \rightarrow B$ is antimonotonic in A and monotonic in B .

$$suc: nat \rightarrow nat$$

function inclusion

$$= nat: \Delta suc \wedge \forall n: nat \cdot suc n: nat$$

definition of suc

$$= nat: nat \wedge \forall n: nat \cdot n+1: nat$$

reflexivity and definition of nat

$$= \top$$

function inclusion

$$f: g = \Delta g: \Delta f \wedge \forall x: \Delta g \cdot fx: gx$$

$$suc: nat \rightarrow nat$$

$$even: int \rightarrow bool$$

$$max: xrat \rightarrow xrat \rightarrow xrat$$

$$A: B \wedge f: B \rightarrow C \wedge C: D \Rightarrow f: A \rightarrow D$$

$$(0, \dots, 10): nat \wedge suc: nat \rightarrow nat \wedge nat: int \Rightarrow suc: (0, \dots, 10) \rightarrow int$$

$$\langle f: (0, \dots, 10) \rightarrow int \cdot \forall n: 0, \dots, 10 \cdot even (f n) \rangle suc$$

$$= \forall n: 0, \dots, 10 \cdot even (suc n)$$

$$= \perp$$

function equality

$$f = g \iff \Delta f = \Delta g \wedge \forall x: \Delta f \cdot fx = gx$$

function composition

If $\neg f: \Delta g$ then

$$\Delta(g f) = \S x: \Delta f \cdot f x: \Delta g$$

$$(g f) x = g (f x)$$

$$\Delta(\text{even suc})$$

$$= \S x: \Delta \text{suc} \cdot \text{suc } x: \Delta \text{even}$$

$$= \S x: \text{nat} \cdot x+1: \text{int}$$

$$= \text{nat}$$

$$(\text{even suc}) 3 = \text{even} (\text{suc } 3) = \text{even } 4 = \top$$

$$(-\text{suc}) 3 = -(\text{suc } 3) = -4$$

$$(\neg \text{even}) 3 = \neg(\text{even } 3) = \neg \perp = \top$$

function composition

Suppose $x, y: int$

$f, g: int \rightarrow int$

$h: int \rightarrow int \rightarrow int$

Then

$$\begin{aligned} & h f x g y \\ = & (((h f) x) g) y \\ = & ((h (f x)) g) y \\ = & (h (f x)) (g y) \\ = & h (f x) (g y) \end{aligned}$$

list as function

If $m: 0, \dots, \#L$ then

$$\underline{\langle n: 0, \dots, \#L \rightarrow Ln \rangle} m = \underline{L m}$$

function \approx list

application \approx indexing

function composition \approx list composition

$$\text{suc } [3; 5; 2] = [4; 6; 3]$$

$$1 \rightarrow 21 \mid [10; 11; 12] = [10; 21; 12]$$

$$\Sigma L = \Sigma n: 0, \dots, \#L \cdot Ln$$

limit

$f: \text{nat} \rightarrow \text{rat}$

$f_0; f_1; f_2; \dots$ is a sequence of rationals

$$(\text{MAX } m \cdot \text{MIN } n \cdot f(m+n)) \leq (\text{LIM } f) \leq (\text{MIN } m \cdot \text{MAX } n \cdot f(m+n))$$

$$\text{LIM } n \cdot 1/(n+1) = 0$$

$$-1 \leq (\text{LIM } n \cdot (-1)^n) \leq 1$$

$$(\text{MIN } f) \leq (\text{LIM } f) \leq (\text{MAX } f)$$

$$\text{xreal} = \text{LIM } (\text{nat} \rightarrow \text{rat})$$

limit

$p: \text{nat} \rightarrow \text{bool}$

$p_0; p_1; p_2; \dots$ is a sequence of booleans

$$\exists m. \forall n. p(m+n) \Rightarrow \text{LIM } p \Rightarrow \forall m. \exists n. p(m+n)$$

$$\exists m. \forall i. i \geq m \Rightarrow p_i \Rightarrow \text{LIM } p$$

$$\exists m. \forall i. i \geq m \Rightarrow \neg p_i \Rightarrow \neg \text{LIM } p$$

$$\text{LIM } n. 1/(n+1) = 0 \quad = \quad \perp$$