Probabilistic Programming

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$\neg x = 1-x \qquad x \wedge y = x \times y \qquad x \vee y = x - x \times y + y$

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distribution value is a probability, sum is 1

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distribution value is a probability, sum is 1

says the frequency of occurrence of values of its variables

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distribution value is a probability, sum is 1

says how strongly we expect or predict each state

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distribution value is a probability, sum is 1

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2^{-n} is a distribution of $n: nat+1$ because $(\forall n: nat+1 \cdot 2^{-n}: prob) \wedge (\Sigma n: nat+1 \cdot 2^{-n}) = 1$

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2^{-n} says $n=3$ with probability $1/8$

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2^{-n-m} is a distribution of $n, m: nat+1$ because

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2^{-n} is a distribution of $n\colon nat+1$ because $(\forall n\colon nat+1\cdot\ 2^{-n}\colon prob) \ \wedge \ (\Sigma n\colon nat+1\cdot\ 2^{-n})=1$

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2^{-n-m} is a distribution of $n, m\colon nat+1$ because

$(\forall n, m\colon nat+1\cdot\ 2^{-n-m}\colon prob) \ \wedge \ (\Sigma n, m\colon nat+1\cdot\ 2^{-n-m})=1$

2^{-n-m} says $n=3 \wedge m=1$ with probability $1/16$

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$n' = n+1$ says: if $n=5$ then $n'=6$

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$(\forall n, n': nat\cdot\ n' = n+1: prob)$

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so (for any value of n) $n' = n+1$ is a one-point distribution of n'

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$n' = n+1$ says: if $n=5$ then $n'=6$ with probability 1

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so (for any value of n) $n' = n+1$ is a one-point distribution of n'

Any implementable deterministic specification is a one-point distribution of the final state.

NonProbabilistic Programming

$ok \quad = \quad (x'=x) \land (y'=y) \land \ldots$

$x{:=}\, e \quad = \quad (x'=e) \land (y'=y) \land \ldots$

if b **then** P **else** Q **fi** $\;=\; b \land P \lor \neg b \land Q$

$$P.Q \;=\; \exists\, x'', y'', \ldots \cdot \;\; (\text{for } x', y', \ldots \text{ substitute } x'', y'', \ldots \text{ in } P)$$
$$\land \;\; (\text{for } x, y, \ldots \text{ substitute } x'', y'', \ldots \text{ in } Q)$$

Probabilistic Programming

$$ok \;=\; (x'=x) \times (y'=y) \times \;...$$

$$x := e \;=\; (x'=e) \times (y'=y) \times \;...$$

$$\textbf{if } b \textbf{ then } P \textbf{ else } Q \textbf{ fi} \;=\; b \times P + (1-b) \times Q$$

$$P.Q \;=\; \Sigma\, x'', y'', ... \cdot \; (\text{for } x', y', ... \text{ substitute } x'', y'', ... \text{ in } P)$$
$$\times \; (\text{for } x, y, ... \text{ substitute } x'', y'', ... \text{ in } Q)$$

example

if 1/3 **then** x:= 0 **else** x:= 1 **fi**

example

if $1/3$ **then** $x:= 0$ **else** $x:= 1$ **fi**

$=$ $1/3 \times (x'=0) + (1 - 1/3) \times (x'=1)$

example

if 1/3 **then** x:= 0 **else** x:= 1 **fi**

$= \quad 1/3 \times (x'=0) + (1-1/3) \times (x'=1)$

evaluate using 0 for x'

$= \quad 1/3 \times (0=0) + (1-1/3) \times (0=1)$

example

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$= \quad 1/3 \times (0=0) + (1 - 1/3) \times (0=1)$

$= \quad 1/3 \times 1 + 2/3 \times 0$

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$$= \quad 1/3$$

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example

if $1/3$ **then** $x := 0$ **else** $x := 1$ **fi**

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example

$$\textbf{if } 1/3 \textbf{ then } x{:=}\ 0 \textbf{ else } x{:=}\ 1 \textbf{ fi}$$

$$= \quad 1/3 \times (x'{=}0) + (1 - 1/3) \times (x'{=}1)$$

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$$= \quad 1/3$$

evaluate using 1 for x'

$$= \quad 1/3 \times (1{=}0) + (1 - 1/3) \times (1{=}1)$$

$$= \quad 1/3 \times 0 + 2/3 \times 1$$

$$= \quad 2/3$$

example

$\textbf{if } 1/3 \textbf{ then } x{:=}\ 0 \textbf{ else } x{:=}\ 1 \textbf{ fi}$

$= \quad 1/3 \times (x'{=}0) + (1 - 1/3) \times (x'{=}1)$

evaluate using 0 for x'

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evaluate using 1 for x'

$= \quad 1/3 \times (1{=}0) + (1 - 1/3) \times (1{=}1)$

$= \quad 1/3 \times 0 + 2/3 \times 1$

$= \quad 2/3$

evaluate using 2 for x'

$= \quad 1/3 \times (2{=}0) + (1 - 1/3) \times (2{=}1)$

example

$$\textbf{if } 1/3 \textbf{ then } x{:=}\ 0 \textbf{ else } x{:=}\ 1 \textbf{ fi}$$

$= \quad 1/3 \times (x'=0) + (1 - 1/3) \times (x'=1)$

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$= \quad 1/3 \times 0 + 2/3 \times 0$

$= \quad 0$

example in one integer variable x

if 1/3 **then** x:= 0 **else** x:= 1 **fi**.

if x=0 **then if** 1/2 **then** x:= x+2 **else** x:= x+3 **fi**

else if 1/4 **then** x:= x+4 **else** x:= x+5 **fi fi**

example in one integer variable x

if 1/3 **then** $x:= 0$ **else** $x:= 1$ **fi**.

if $x=0$ **then if** 1/2 **then** $x:= x+2$ **else** $x:= x+3$ **fi**

else if 1/4 **then** $x:= x+4$ **else** $x:= x+5$ **fi fi**

$$= \quad \Sigma x''\cdot \quad ((x''=0)/3 + (x''=1)\times 2/3)$$

$$\times \quad (\quad (x''=0) \times ((x'=x''+2)/2 + (x'=x''+3)/2)$$

$$+ \quad (x''\neq 0) \times ((x'=x''+4)/4 + (x'=x''+5)\times 3/4))$$

example in one integer variable x

if $1/3$ **then** $x{:=}\ 0$ **else** $x{:=}\ 1$ **fi**. ⟵

if $x{=}0$ **then if** $1/2$ **then** $x{:=}\ x{+}2$ **else** $x{:=}\ x{+}3$ **fi**

else if $1/4$ **then** $x{:=}\ x{+}4$ **else** $x{:=}\ x{+}5$ **fi fi**

$$
\begin{aligned}
= \quad \Sigma x''\cdot \quad & ((x''{=}0)/3 + (x''{=}1)\times 2/3) \quad \longleftarrow \\
\times \quad & (\ \ (x''{=}0) \times ((x'{=}x''{+}2)/2 + (x'{=}x''{+}3)/2) \\
& +\ (x''{\neq}0) \times ((x'{=}x''{+}4)/4 + (x'{=}x''{+}5)\times 3/4))
\end{aligned}
$$

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$$= \quad (x'=2)/6 + (x'=3)/6 + (x'=5)/6 + (x'=6)/2$$

Average

after P, average value of e is $P.e$

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as n varies over $nat+1$ according to distribution 2^{-n} the average value of n^2 is

$2^{-n'}.\ n^2$

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as n varies over $nat{+}1$ according to distribution 2^{-n} the average value of n^2 is

$$2^{-n'}.\ n^2$$

$$= \quad \Sigma n'' \colon nat{+}1 \cdot\ 2^{-n''} \times n''^2$$

$$= \quad 6$$

Average

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after **if** 1/3 **then** x:= 0 **else** x:= 1 **fi**.

if x=0 **then if** 1/2 **then** x:= x+2 **else** x:= x+3 **fi**

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x

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else if $1/4$ **then** $x{:=}\ x{+}4$ **else** $x{:=}\ x{+}5$ **fi fi**.

x

$=\quad (x'{=}2)/6 + (x'{=}3)/6 + (x'{=}5)/6 + (x'{=}6)/2.\ \ x$

Average

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x

$= \quad (x'=2)/6 + (x'=3)/6 + (x'=5)/6 + (x'=6)/2.\ x$

$= \quad \Sigma x''\cdot\ ((x''=2)/6 + (x''=3)/6 + (x''=5)/6 + (x''=6)/2) \times x''$

Average

after P, average value of e is $P.\ e$

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x

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$= \quad \Sigma x''\cdot\ ((x''{=}2)/6 + (x''{=}3)/6 + (x''{=}5)/6 + (x''{=}6)/2) \times x''$

$= \quad 1/6 \times 2 + 1/6 \times 3 + 1/6 \times 5 + 1/2 \times 6$

Average

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$= \quad 1/6 \times 2 + 1/6 \times 3 + 1/6 \times 5 + 1/2 \times 6$

$= \quad 4 + 2/3$

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after P , average value of e is $P.\, e$

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after P, average value of e is $P.\,e$

after P, probability that b is true is $P.\,b$

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Probability is just the average value of a binary expression.

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$x>3$

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Probability is just the average value of a binary expression.

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else if 1/4 **then** $x:= x+4$ **else** $x:= x+5$ **fi fi**.

$x>3$

$= \quad (x'=2)/6 + (x'=3)/6 + (x'=5)/6 + (x'=6)/2.\ \ x>3$

$= \quad \Sigma x''\cdot\ ((x''=2)/6 + (x''=3)/6 + (x''=5)/6 + (x''=6)/2) \times (x''>3)$

$= \quad 1/6 \times (2>3) + 1/6 \times (3>3) + 1/6 \times (5>3) + 1/2 \times (6>3)$

$= \quad 2/3$

Random Number Generator

rand n

Random Number Generator

rand n

x=x therefore *rand n* = *rand n* ?

Random Number Generator

rand n

$x = x$ therefore $rand\ n = rand\ n$ **?**

$x + x = 2 \times x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x=x$ therefore $rand\ n = rand\ n$ **?**

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Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x=x$ therefore $rand\ n = rand\ n$ **?**

$x+x = 2 \times x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

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Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x{=}x$ therefore $rand\ n = rand\ n$ **?**

$x{+}x = 2{\times}x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

$$x{:=}\ rand\ 2.\ \ x{:=}\ x + rand\ 3$$

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x=x$ therefore $rand\ n = rand\ n$ **?**

$x+x = 2{\times}x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

$x:= rand\ 2.\ \ x:= x + rand\ 3$ replace one $rand$ with r and one with s

$=$ $\Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x:= r)/2.\ \ (x:= x + s)/3$

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x{=}x$ therefore $rand\ n = rand\ n$ **?**

$x{+}x = 2{\times}x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

$x{:=}\ rand\ 2.\ \ x{:=}\ x + rand\ 3$ — replace one $rand$ with r and one with s

$= \quad \Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x{:=}\ r)/2.\ \ (x{:=}\ x + s)/3$ — Substitution Law

$= \quad \Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x' = r{+}s) / 6$

Random Number Generator

rand n has value *r* with probability $(r: 0,..n) / n$

$x=x$ therefore $\mathit{rand}\ n = \mathit{rand}\ n$ **?**

$x+x = 2\times x$ therefore $\mathit{rand}\ n + \mathit{rand}\ n = 2 \times \mathit{rand}\ n$ **?**

Replace *rand n* with r: *int* with distribution $(r: 0,..n) / n$

Replace *rand n* with $r: 0,..n$ with distribution $1/n$

	$x{:=}\ \mathit{rand}\ 2.\ \ x{:=}\ x + \mathit{rand}\ 3$	replace one *rand* with r and one with s
=	$\Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x{:=}\ r)/2.\ \ (x{:=}\ x + s)/3$	Substitution Law
=	$\Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x' = r+s)\ /\ 6$	sum
=	$((x' = 0+0) + (x' = 0+1) + (x' = 0+2) + (x' = 1+0) + (x' = 1+1) + (x' = 1+2))\ /\ 6$	

Random Number Generator

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$=\ \Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x:= r)/2.\ \ (x:= x + s)/3$ Substitution Law

$=\ \Sigma r: 0,..2\cdot\ \Sigma s: 0,..3\cdot\ (x' = r+s) / 6$ sum

$=\ ((x' = 0+0) + (x' = 0+1) + (x' = 0+2) + (x' = 1+0) + (x' = 1+1) + (x' = 1+2)) / 6$

$=\ (x'=0) / 6 + (x'=1) / 3 + (x'=2) / 3 + (x'=3) / 6$

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x=x$ therefore $rand\ n = rand\ n$ **?**

$x+x = 2\times x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

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Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

$$x:= rand\ 2.\ \ x:= x + rand\ 3$$

Random Number Generator

rand n has value *r* with probability $(r: 0,..n) / n$

$x{=}x$ therefore $rand\ n = rand\ n$ **?**

$x{+}x = 2{\times}x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace *rand n* with r: *int* with distribution $(r: 0,..n) / n$

Replace *rand n* with r: $0,..n$ with distribution $1/n$

$$x{:=}\ rand\ 2.\ \ x{:=}\ x + rand\ 3$$ replace *rand*

$$=\quad (x'{:}\ 0,..2)/2.\ \ (x'{:}\ x{+}(0,..3))/3$$

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x{=}x$ therefore $rand\ n = rand\ n$ **?**

$x{+}x = 2{\times}x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

	$x{:=}\ rand\ 2.\ \ x{:=}\ x + rand\ 3$	replace *rand*
$=$	$(x'{:}\ 0,..2)/2.\ \ (x'{:}\ x{+}(0,..3))/3$	sequential composition
$=$	$\Sigma x''\cdot\ (x''{:}\ 0,..2)/2 \times (x'{:}\ x''{+}(0,..3))/3$	

Random Number Generator

$rand\ n$ has value r with probability $(r{:}\ 0,..n)\ /\ n$

$x{=}x$ therefore $rand\ n = rand\ n$ **?**

$x{+}x = 2{\times}x$ therefore $rand\ n + rand\ n\ =\ 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r{:}\ int$ with distribution $(r{:}\ 0,..n)\ /\ n$

Replace $rand\ n$ with $r{:}\ 0,..n$ with distribution $1/n$

	$x{:=}\ rand\ 2.\ \ x{:=}\ x + rand\ 3$	replace *rand*
=	$(x'{:}\ 0,..2)/2.\ \ (x'{:}\ x{+}(0,..3))/3$	sequential composition
=	$\Sigma x''\cdot\ (x''{:}\ 0,..2)/2 \times (x'{:}\ x''{+}(0,..3))/3$	sum
=	$1/2 \times (x'{:}\ 0,..3)/3\ +\ 1/2 \times (x'{:}\ 1,..4)/3$	

Random Number Generator

$rand\ n$ has value r with probability $(r: 0,..n) / n$

$x=x$ therefore $rand\ n = rand\ n$ **?**

$x+x = 2{\times}x$ therefore $rand\ n + rand\ n = 2 \times rand\ n$ **?**

Replace $rand\ n$ with $r: int$ with distribution $(r: 0,..n) / n$

Replace $rand\ n$ with $r: 0,..n$ with distribution $1/n$

	$x:= rand\ 2.\ \ x:= x + rand\ 3$	replace $rand$
$=$	$(x': 0,..2)/2.\ \ (x': x+(0,..3))/3$	sequential composition
$=$	$\Sigma x''\cdot\ (x'': 0,..2)/2 \times (x': x''+(0,..3))/3$	sum
$=$	$1/2 \times (x': 0,..3)/3 + 1/2 \times (x': 1,..4)/3$	
$=$	$(x'=0) / 6 + (x'=1) / 3 + (x'=2) / 3 + (x'=3) / 6$	

Blackjack

You are dealt a card from a deck; its value is in the range 1 to 13 inclusive. You may stop with just one card, or have a second card if you want. Your object is to get a total as near as possible to 14, but not over 14. Your strategy is to take a second card if the first is under 7.

Blackjack

You are dealt a card from a deck; its value is in the range 1 to 13 inclusive. You may stop with just one card, or have a second card if you want. Your object is to get a total as near as possible to 14 , but not over 14 . Your strategy is to take a second card if the first is under 7 .

$x{:=}\ (rand\ 13) + 1.\ \ \mathbf{if}\ x{<}7\ \mathbf{then}\ x{:=}\ x + (rand\ 13) + 1\ \mathbf{else}\ ok\ \mathbf{fi}$

Blackjack

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$x{:=}\ (\mathit{rand}\ 13) + 1.\ \ \mathbf{if}\ x{<}7\ \mathbf{then}\ x{:=}\ x + (\mathit{rand}\ 13) + 1\ \mathbf{else}\ \mathit{ok}\ \mathbf{fi}$ replace *rand* and *ok*

$=\quad (x'\text{: } (0,..13){+}1)/13.\ \ \mathbf{if}\ x{<}7\ \mathbf{then}\ (x'\text{: } x{+}(0,..13){+}1)/13\ \mathbf{else}\ x'{=}x\ \mathbf{fi}$

Blackjack

You are dealt a card from a deck; its value is in the range 1 to 13 inclusive. You may stop with just one card, or have a second card if you want. Your object is to get a total as near as possible to 14 , but not over 14 . Your strategy is to take a second card if the first is under 7 .

$x:= (rand\ 13) + 1.\ \textbf{if}\ x<7\ \textbf{then}\ x:= x + (rand\ 13) + 1\ \textbf{else}\ ok\ \textbf{fi}$ replace *rand* and *ok*

$=\quad (x'\colon (0,..13)+1)/13.\ \textbf{if}\ x<7\ \textbf{then}\ (x'\colon x+(0,..13)+1)/13\ \textbf{else}\ x'=x\ \textbf{fi}$ replace . and **if**

$=\quad \Sigma x''\cdot\ (x''\colon 1,..14)/13 \times ((x''<7)\times(x'\colon x''+1,..x''+14)/13 + (x''\geq 7)\times(x'=x''))$

Blackjack

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$x{:=}\ (\mathit{rand}\ 13) + 1.\ \ \mathbf{if}\ x{<}7\ \mathbf{then}\ x{:=}\ x + (\mathit{rand}\ 13) + 1\ \mathbf{else}\ \mathit{ok}\ \mathbf{fi}$ replace *rand* and *ok*

$= (x'\!: (0,..13){+}1)/13.\ \ \mathbf{if}\ x{<}7\ \mathbf{then}\ (x'\!: x{+}(0,..13){+}1)/13\ \mathbf{else}\ x'{=}x\ \mathbf{fi}$ replace . and **if**

$= \Sigma x''\cdot\ \ (x''\!: 1,..14)/13 \times ((x''{<}7){\times}(x'\!: x''{+}1,..x''{+}14)/13 + (x''{\geq}7){\times}(x'{=}x''))$

by several omitted steps

$= ((2{\leq}x'{<}7){\times}(x'{-}1) + (7{\leq}x'{<}14){\times}19 + (14{\leq}x'{<}20){\times}(20{-}x'))\ /\ 169$

Player x plays "under n" and player y plays "under $n+1$"

Player x plays "under n" and player y plays "under n+1"

c:= ($rand$ 13) + 1. d:= ($rand$ 13) + 1

Player x plays "under n" and player y plays "under n+1"

c:= ($rand$ 13) + 1. d:= ($rand$ 13) + 1.

if $c < n$ **then** x:=c+d **else** x:=c **fi**

Player x plays "under n" and player y plays "under n+1"

c:= (*rand* 13) + 1. d:= (*rand* 13) + 1.

if $c < n$ **then** x:=c+d **else** x:=c **fi**. **if** $c < n$+1 **then** y:=c+d **else** y:=c **fi**

Player x plays "under n" and player y plays "under n+1"

c:= ($rand$ 13) + 1. d:= ($rand$ 13) + 1.

if $c < n$ **then** x:=c+d **else** x:=c **fi**. **if** $c < n$+1 **then** y:=c+d **else** y:=c **fi**.

$y<x\le 14 \lor x\le 14<y$

Player x plays "under n" and player y plays "under $n+1$"

$c:= (rand\ 13) + 1.\ \ d:= (rand\ 13) + 1.$

if $c < n$ **then** $x:=c+d$ **else** $x:=c$ **fi**. **if** $c < n+1$ **then** $y:=c+d$ **else** $y:=c$ **fi**.

$y<x\le 14 \lor x\le 14<y$

replace *rand*

$=$ $(c'\!: (0,..13)+1 \ \land\ d'\!: (0,..13)+1 \ \land\ x'=x \ \land\ y'=y) / 13 / 13.$

if $c < n$ **then** $x:=c+d$ **else** $x:=c$ **fi**. **if** $c < n+1$ **then** $y:=c+d$ **else** $y:=c$ **fi**.

$y<x\le 14 \lor x\le 14<y$

Player x plays "under n" and player y plays "under $n+1$"

$c := (rand\ 13) + 1.\ \ d := (rand\ 13) + 1.$

if $c < n$ **then** $x:=c+d$ **else** $x:=c$ **fi**. **if** $c < n+1$ **then** $y:=c+d$ **else** $y:=c$ **fi**.

$y<x\le 14 \lor x\le 14<y$

replace *rand*

$=$ $(c'\!: (0,..13)+1 \ \land\ d'\!: (0,..13)+1 \ \land\ x'=x \ \land\ y'=y)\ /\ 13\ /\ 13.$

if $c < n$ **then** $x:=c+d$ **else** $x:=c$ **fi**. **if** $c < n+1$ **then** $y:=c+d$ **else** $y:=c$ **fi**.

$y<x\le 14 \lor x\le 14<y$

4 omitted steps

$=$ $(n-1)\ /\ 169$

Player x plays "under n" and player y plays "under $n+1$"

$c := (rand\ 13) + 1.\ \ d := (rand\ 13) + 1.$

if $c < n$ **then** $x:=c+d$ **else** $x:=c$ **fi**. **if** $c < n+1$ **then** $y:=c+d$ **else** $y:=c$ **fi**.

$y<x\le 14 \lor x\le 14<y$

replace *rand*

$=$ $(c'\!: (0,..13)+1 \ \land\ d'\!: (0,..13)+1 \ \land\ x'=x \ \land\ y'=y) / 13 / 13.$

if $c < n$ **then** $x:=c+d$ **else** $x:=c$ **fi**. **if** $c < n+1$ **then** $y:=c+d$ **else** $y:=c$ **fi**.

$y<x\le 14 \lor x\le 14<y$

4 omitted steps

$=$ $(n-1) / 169$

probability that x wins is $(n-1) / 169$

probability that y wins is $(14-n) / 169$

probability of a tie is $12/13$

Player x plays “under n ” and player y plays “under n+1 ”

c:= ($rand$ 13) + 1. d:= ($rand$ 13) + 1.

if $c < n$ **then** x:=c+d **else** x:=c **fi**. **if** $c < n$+1 **then** y:=c+d **else** y:=c **fi**.

$y<x\le 14 \lor x\le 14<y$

replace *rand*

= (c': (0,..13)+1 ∧ d': (0,..13)+1 ∧ x'=x ∧ y'=y) / 13 / 13.

if $c < n$ **then** x:=c+d **else** x:=c **fi**. **if** $c < n$+1 **then** y:=c+d **else** y:=c **fi**.

$y<x\le 14 \lor x\le 14<y$

4 omitted steps

= (n–1) / 169

probability that x wins is (n–1) / 169

probability that y wins is (14–n) / 169

probability of a tie is 12/13

“under 8 ” beats both “under 7 ” and “under 9 ”

Dice

If you repeatedly throw a pair of six-sided dice until they are equal, how long does it take?

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$$R \;\Leftarrow\; u{:=}\ (rand\ 6) + 1.\ \ v{:=}\ (rand\ 6) + 1.\ \ \textbf{if}\ u{=}v\ \textbf{then}\ ok\ \textbf{else}\ t{:=}\ t{+}1.\ \ R\ \textbf{fi}$$

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hypothesis: t' has the distribution $(t' \geq t) \times (5/6)^{t'-t} \times 1/6$

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$u{:=}\,(rand\ 6)+1.\ \ v{:=}\,(rand\ 6)+1.$

$\mathbf{if}\ u{=}v\ \mathbf{then}\ t'{=}t\ \mathbf{else}\ t{:=}\,t{+}1.\ \ (t'{\geq}t)\times(5/6)^{t'-t}\times 1/6\ \mathbf{fi}$

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$$R \Leftarrow u:= (rand\ 6) + 1.\ v:= (rand\ 6) + 1.\ \mathbf{if}\ u=v\ \mathbf{then}\ ok\ \mathbf{else}\ t:= t+1.\ R\ \mathbf{fi}$$

$u:= (rand\ 6) + 1.\ v:= (rand\ 6) + 1.$ — replace *rand*

$\mathbf{if}\ u=v\ \mathbf{then}\ t'=t\ \mathbf{else}\ t:= t+1.\ (t'\ge t) \times (5/6)^{t'-t} \times 1/6\ \mathbf{fi}$ — Substitution Law

$=\quad (u'\!: 1,..7\ \land\ v'=v\ \land\ t'=t)/6.\ (u'=u\ \land\ v'\!: 1,..7\ \land\ t'=t)/6.$

$\mathbf{if}\ u=v\ \mathbf{then}\ t'=t\ \mathbf{else}\ (t'\ge t+1) \times (5/6)^{t'-t-1} / 6\ \mathbf{fi}$

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$u{:=}\,(rand\ 6)+1.\ \ v{:=}\,(rand\ 6)+1.$ — replace *rand*

$\textbf{if } u{=}v \textbf{ then } t'{=}t \textbf{ else } t{:=}\,t{+}1.\ \ (t'{\ge}t)\times(5/6)^{t'-t}\times 1/6\ \textbf{fi}$ — Substitution Law

$=\quad (u'{:}\ 1,..7\ \wedge\ v'{=}v\ \wedge\ t'{=}t)/6.\ \ (u'{=}u\ \wedge\ v'{:}\ 1,..7\ \wedge\ t'{=}t)/6.$ — replace first .

$\textbf{if } u{=}v \textbf{ then } t'{=}t \textbf{ else } (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\ /\ 6\ \textbf{fi}$

$=\quad (u', v'{:}\ 1,..7\ \wedge\ t'{=}t)/36.$

$\textbf{if } u{=}v \textbf{ then } t'{=}t \textbf{ else } (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\ /\ 6\ \textbf{fi}$

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If you repeatedly throw a pair of six-sided dice until they are equal, how long does it take?

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$u{:=}\,(\mathit{rand}\ 6)+1.\ \ v{:=}\,(\mathit{rand}\ 6)+1.$ — replace *rand*

$\textbf{if}\ u{=}v\ \textbf{then}\ t'{=}t\ \textbf{else}\ t{:=}\,t{+}1.\ \ (t'{\ge}t)\times(5/6)^{t'-t}\times 1/6\ \textbf{fi}$ — Substitution Law

$=\quad (u'{:}\ 1,..7\ \wedge\ v'{=}v\ \wedge\ t'{=}t)/6.\ \ (u'{=}u\ \wedge\ v'{:}\ 1,..7\ \wedge\ t'{=}t)/6.$ — replace first .

$\textbf{if}\ u{=}v\ \textbf{then}\ t'{=}t\ \textbf{else}\ (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\,/\,6\ \textbf{fi}$

$=\quad (u',v'{:}\ 1,..7\ \wedge\ t'{=}t)/36.$ — replace .

$\textbf{if}\ u{=}v\ \textbf{then}\ t'{=}t\ \textbf{else}\ (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\,/\,6\ \textbf{fi}$ — replace **if**

$$=\quad \Sigma u'',v''{:}\ 1,..7\cdot\ \Sigma t''\cdot\ (t''{=}t)/36\times(\ (u''{=}v'')\times(t'{=}t'')$$

$$+\ (u''{\ne}v'')\times(t'{\ge}t''{+}1)\times(5/6)^{t'-t''-1}\,/\,6)$$

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$u{:=}\ (rand\ 6)+1.\ \ v{:=}\ (rand\ 6)+1.$ — replace *rand*

$\textbf{if}\ u{=}v\ \textbf{then}\ t'{=}t\ \textbf{else}\ t{:=}\ t{+}1.\ \ (t'{\ge}t)\times(5/6)^{t'-t}\times 1/6\ \textbf{fi}$ — Substitution Law

$=\quad (u'{:}\ 1,..7\ \wedge\ v'{=}v\ \wedge\ t'{=}t)/6.\ \ (u'{=}u\ \wedge\ v'{:}\ 1,..7\ \wedge\ t'{=}t)/6.$ — replace first .

$\textbf{if}\ u{=}v\ \textbf{then}\ t'{=}t\ \textbf{else}\ (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\ /\ 6\ \textbf{fi}$

$=\quad (u',v'{:}\ 1,..7\ \wedge\ t'{=}t)/36.$ — replace .

$\textbf{if}\ u{=}v\ \textbf{then}\ t'{=}t\ \textbf{else}\ (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\ /\ 6\ \textbf{fi}$ — replace **if**

$=\quad \Sigma u'',v''{:}\ 1,..7\cdot\ \Sigma t''\cdot\ \ (t''{=}t)/36\times(\ \ (u''{=}v'')\times(t'{=}t'')$

$\qquad +\ (u''{\ne}v'')\times(t'{\ge}t''{+}1)\times(5/6)^{t'-t''-1}\ /\ 6)$ — sum

$=\quad (6\times(t'{=}t)\ +\ 30\times(t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\ /\ 6)\ /\ 36$

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	$u{:=}\,(\mathit{rand}\ 6)+1.\;\; v{:=}\,(\mathit{rand}\ 6)+1.$	replace *rand*
	$\mathbf{if}\ u{=}v\ \mathbf{then}\ t'{=}t\ \mathbf{else}\ t{:=}\,t{+}1.\;\; (t'{\ge}t)\times(5/6)^{t'-t}\times 1/6\ \mathbf{fi}$	Substitution Law
$=$	$(u'{:}\ 1,..7\ \wedge\ v'{=}v\ \wedge\ t'{=}t)/6.\;\; (u'{=}u\ \wedge\ v'{:}\ 1,..7\ \wedge\ t'{=}t)/6.$	replace first .
	$\mathbf{if}\ u{=}v\ \mathbf{then}\ t'{=}t\ \mathbf{else}\ (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\,/\,6\ \mathbf{fi}$	
$=$	$(u', v'{:}\ 1,..7\ \wedge\ t'{=}t)/36.$	replace .
	$\mathbf{if}\ u{=}v\ \mathbf{then}\ t'{=}t\ \mathbf{else}\ (t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\,/\,6\ \mathbf{fi}$	replace **if**
$=$	$\Sigma u'', v''{:}\ 1,..7\cdot\ \Sigma t''\cdot\ (t''{=}t)/36\times(\ (u''{=}v'')\times(t'{=}t'')$	
	$+\ (u''{\ne}v'')\times(t'{\ge}t''{+}1)\times(5/6)^{t'-t''-1}\,/\,6)$	sum
$=$	$(6\times(t'{=}t)\ +\ 30\times(t'{\ge}t{+}1)\times(5/6)^{t'-t-1}\,/\,6)\,/\,36$	combine
$=$	$(t'{\ge}t)\times(5/6)^{t'-t}\times 1/6$	

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The average value of t' is $(t' \geq t) \times (5/6)^{t'-t} \times 1/6.\ t$

Dice

If you repeatedly throw a pair of six-sided dice until they are equal, how long does it take?

$$R \;\Leftarrow\; u{:=}\,(rand\ 6)+1.\;\; v{:=}\,(rand\ 6)+1.\;\; \mathbf{if}\ u{=}v\ \mathbf{then}\ ok\ \mathbf{else}\ t{:=}\,t{+}1.\;\; R\ \mathbf{fi}$$

The average value of t' is $(t'{\geq}t) \times (5/6)^{t'-t} \times 1/6.\; t \quad = \quad t{+}5$