

Monotonicity and Antimonotonicity

Monotonicity and Antimonotonicity

covariance and contravariance

Monotonicity and Antimonotonicity

covariance and contravariance
varies directly as and varies inversely as

Monotonicity and Antimonotonicity

covariance **and** **contravariance**
varies directly as **and** **varies inversely as**
nondecreasing **and** **nonincreasing**

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varies directly as **and** **varies inversely as**
nondecreasing **and** **nonincreasing**
sorted **and** **sorted backwards**

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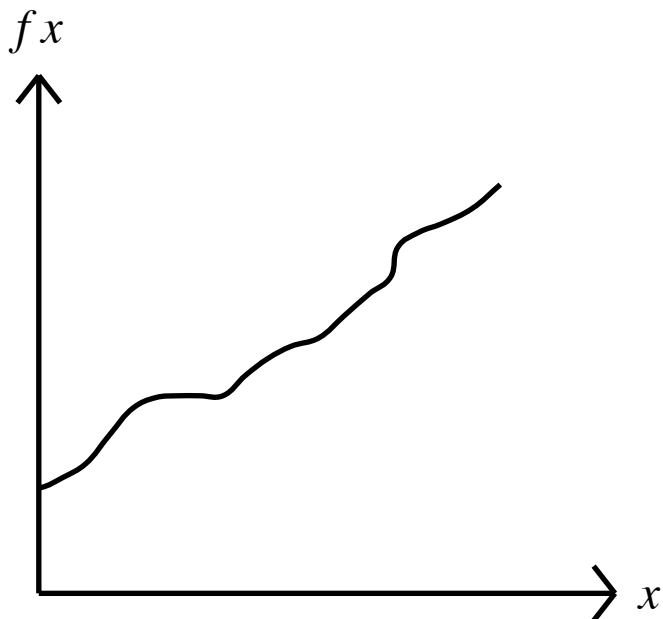
$$x \leq y \Rightarrow f x \leq f y$$

$$x \leq y \Rightarrow f x \geq f y$$

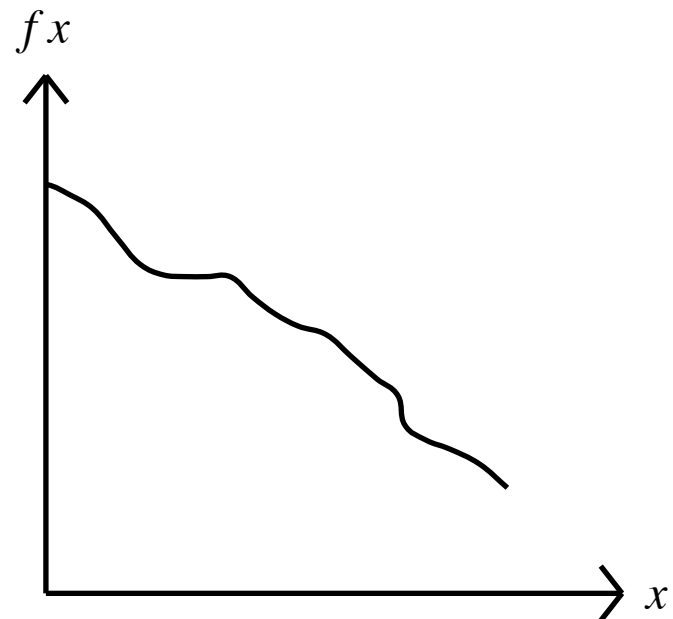
Monotonicity and Antimonotonicity

covariance and **contravariance**
varies directly as and **varies inversely as**
nondecreasing and **nonincreasing**
sorted and **sorted backwards**

$$x \leq y \Rightarrow f(x) \leq f(y)$$



$$x \leq y \Rightarrow f(x) \geq f(y)$$



Monotonicity and Antimonotonicity

number:

binary:

Monotonicity and Antimonotonicity

number: $x \leq y$

binary: $x \Rightarrow y$

Monotonicity and Antimonotonicity

number: $x \leq y$ x is less than or equal to y

binary: $x \Rightarrow y$ x implies y

Monotonicity and Antimonotonicity

number: $x \leq y$ x is less than or equal to y

binary: $x \Rightarrow y$ x implies y x is false or y is true

Monotonicity and Antimonotonicity

number: $x \leq y$ x is less than or equal to y

binary: $x \Rightarrow y$ x implies y x is stronger than or equal to y

Monotonicity and Antimonotonicity

number: $x \leq y$ x is less than or equal to y
 $-\infty \leq +\infty$ $0 \leq 1$ smaller \leq larger

binary: $x \Rightarrow y$ x implies y x is stronger than or equal to y
 $\perp \Rightarrow \top$ stronger \Rightarrow weaker

Monotonicity and Antimonotonicity

number: $x \leq y$ x is less than or equal to y
 $-\infty \leq +\infty$ $0 \leq 1$ smaller \leq larger
 $x \leq y \Rightarrow f x \leq f y$ f is monotonic

binary: $x \Rightarrow y$ x implies y x is stronger than or equal to y
 $\perp \Rightarrow \top$ stronger \Rightarrow weaker
 $x \Rightarrow y \Rightarrow f x \Rightarrow f y$ f is monotonic

Monotonicity and Antimonotonicity

number:

$$x \leq y$$

x is less than or equal to y

$$-\infty \leq +\infty \quad 0 \leq 1$$

smaller \leq larger

$$x \leq y \Rightarrow f x \leq f y$$

f is monotonic

as x gets larger, $f x$ gets larger (or equal)

binary:

$$x \Rightarrow y$$

x implies y

x is stronger than or equal to y

$$\perp \Rightarrow \top$$

stronger \Rightarrow weaker

$$x \Rightarrow y \Rightarrow f x \Rightarrow f y$$

f is monotonic

as x gets weaker, $f x$ gets weaker (or equal)

Monotonicity and Antimonotonicity

number:	$x \leq y$	x is less than or equal to y
	$-\infty \leq +\infty \quad 0 \leq 1$	smaller \leq larger
	$x \leq y \Rightarrow f x \leq f y$	f is monotonic as x gets larger, $f x$ gets larger (or equal)
	$x \leq y \Rightarrow f x \geq f y$	f is antimonotonic as x gets larger, $f x$ gets smaller (or equal)
binary:	$x \Rightarrow y$	x implies y x is stronger than or equal to y
	$\perp \Rightarrow \top$	stronger \Rightarrow weaker
	$x \Rightarrow y \Rightarrow f x \Rightarrow f y$	f is monotonic as x gets weaker, $f x$ gets weaker (or equal)
	$x \Rightarrow y \Rightarrow f x \Leftarrow f y$	f is antimonotonic as x gets weaker, $f x$ gets stronger (or equal)

Monotonicity and Antimonotonicity

$\neg a$

antimonotonic in a

Monotonicity and Antimonotonicity

$\neg a$ antimonotonic in a

$a \wedge b$ monotonic in a monotonic in b

Monotonicity and Antimonotonicity

$\neg a$ antimonotonic in a

$a \wedge b$ monotonic in a monotonic in b

$a \vee b$ monotonic in a monotonic in b

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
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$a \wedge b$ monotonic in a monotonic in b

$a \vee b$ monotonic in a monotonic in b

$a \Rightarrow b$ antimonotonic in a monotonic in b

$a \Leftarrow b$ monotonic in a antimonotonic in b

if a then b else c fi monotonic in b monotonic in c

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$\neg(a \wedge \neg(a \vee b))$

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if a then b else c fi monotonic in b monotonic in c

$\neg(a \wedge \neg(a \vee b))$

$\neg(a \wedge \neg \overline{a})$



use the Law of Generalization $a \Rightarrow a \vee b$

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$a \Leftarrow b$ monotonic in a antimonotonic in b

if a then b else c fi monotonic in b monotonic in c

$$\neg(a \wedge \underline{\neg(a \vee b)})$$



$$\neg(a \wedge \underline{\neg a})$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a then b else c fi	monotonic in b	monotonic in c

$$\frac{\neg(a \wedge \neg(a \vee b))}{\neg(a \wedge \neg a)}$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

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if a then b else c fi	monotonic in b	monotonic in c

$$\frac{\neg(a \wedge \neg(a \vee b))}{\neg(a \wedge \neg a)}$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

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$a \wedge b$ monotonic in a monotonic in b

$a \vee b$ monotonic in a monotonic in b

$a \Rightarrow b$ antimonotonic in a monotonic in b

$a \Leftarrow b$ monotonic in a antimonotonic in b

if a then b else c fi monotonic in b monotonic in c

$\neg(a \wedge \neg(a \vee b))$ use the Law of Generalization $a \Rightarrow a \vee b$

$\Leftarrow \neg(a \wedge \neg a)$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a then b else c fi	monotonic in b	monotonic in c

$$\neg(a \wedge \neg(a \vee b))$$

use the Law of Generalization $a \Rightarrow a \vee b$

$$\Leftarrow \neg(a \wedge \neg a)$$

now use the Law of Noncontradiction

$$= \top$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

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In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\neg(\underline{a \wedge \neg(avb)})$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{array}{l} \neg(a \wedge \neg(avb)) \\ = \quad \neg(a \wedge \neg(\top \vee b)) \end{array} \quad \text{context } a \text{ in } \neg(avb)$$


Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg \top) \end{aligned}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg\top) && \text{Theorem Table for } \neg \\ = & \neg(a \wedge \perp) \end{aligned}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg\top) && \text{Theorem Table for } \neg \\ = & \neg(a \wedge \perp) && \text{Base Law for } \wedge \\ = & \neg\perp \end{aligned}$$

Context

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$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg \top) && \text{Theorem Table for } \neg \\ = & \neg(a \wedge \perp) && \text{Base Law for } \wedge \\ = & \neg \perp && \text{Binary Axiom, or Theorem Table for } \neg \\ = & \top && \end{aligned}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg \top) && \text{Theorem Table for } \neg \\ = & \neg(a \wedge \perp) && \text{Base Law for } \wedge \\ = & \neg \perp && \text{Binary Axiom, or Theorem Table for } \neg \\ = & \top \end{aligned}$$

If a is a theorem, then we can replace it with \top .

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg \top) && \text{Theorem Table for } \neg \\ = & \neg(a \wedge \perp) && \text{Base Law for } \wedge \\ = & \neg \perp && \text{Binary Axiom, or Theorem Table for } \neg \\ = & \top \end{aligned}$$

If a is a theorem, then we can replace it with \top .

If a is an antitheorem, then $a \wedge \text{anything}$ is an antitheorem.

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

$$\begin{aligned} & \neg(a \wedge \neg(avb)) && \text{context } a \text{ in } \neg(avb) \\ = & \neg(a \wedge \neg(\top \vee b)) && \text{Symmetry Law and Base Law for } \vee \\ = & \neg(a \wedge \neg\top) && \text{Theorem Table for } \neg \\ = & \neg(a \wedge \perp) && \text{Base Law for } \wedge \\ = & \neg\perp && \text{Binary Axiom, or Theorem Table for } \neg \\ = & \top && \\ \\ & x=y \wedge x=z && \text{context } x=y \text{ in } x=z \\ = & x=y \wedge y=z && \end{aligned}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

In $exp0 \wedge exp1$, $exp1$ is a local axiom within $exp0$.

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$$= \begin{array}{c} a \wedge a \\ \downarrow \\ \top \wedge a \end{array} \quad \text{use right conjunct as context in left conjunct}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

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$$= \begin{array}{c} a \wedge a \\ a \wedge \downarrow \\ a \wedge \top \end{array} \quad \text{use left conjunct as context in right conjunct}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

In $exp0 \wedge exp1$, $exp1$ is a local axiom within $exp0$.

$$= \begin{array}{c} a \wedge a \\ \downarrow \\ \top \wedge a \end{array} \quad \text{use right conjunct as context in left conjunct}$$

$$= \begin{array}{c} a \wedge a \\ a \wedge \downarrow \\ \top \end{array} \quad \text{use left conjunct as context in right conjunct}$$

$$= \begin{array}{c} a \wedge a \\ \downarrow \quad \downarrow \\ \top \wedge \top \end{array} \quad \text{X}$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

In $exp0 \wedge exp1$, $exp1$ is a local axiom within $exp0$.

In $exp0 \vee exp1$, $exp0$ is a local anti-axiom within $exp1$.

In $exp0 \vee exp1$, $exp1$ is a local anti-axiom within $exp0$.

In $exp0 \Rightarrow exp1$, $exp0$ is a local axiom within $exp1$.

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In $exp0 \Leftarrow exp1$, $exp0$ is a local anti-axiom within $exp1$.

In $exp0 \Leftarrow exp1$, $exp1$ is a local axiom within $exp0$.

In **if** $exp0$ **then** $exp1$ **else** $exp2$ **fi** , $exp0$ is a local axiom within $exp1$.

In **if** $exp0$ **then** $exp1$ **else** $exp2$ **fi** , $exp0$ is a local anti-axiom within $exp2$.

In **if** $exp0$ **then** $exp1$ **else** $exp2$ **fi** , $exp1=exp2$ is a local anti-axiom within $exp0$.

Number Theory

number expressions represent quantity

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number expressions

0 1 2 597 1.2 1e10 ∞

$-x$ $x+y$ $x-y$ $x \times y$ x/y $\frac{x}{y}$ x^y $x \uparrow y$ $x \downarrow y$

if a then x else y fi

Number Theory

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$-x$ $x+y$ $x-y$ $x \times y$ x/y $\frac{x}{y}$ x^y $x \uparrow y$ $x \downarrow y$

if a then x else y fi

binary expressions

$x=y$ $x \neq y$ $x < y$ $x > y$ $x \leq y$ $x \geq y$

Character Theory

“A” “a” “ ” ““““”” “”””””

succ *pred* **if then else fi**

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Character Theory

“A”

“a”

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succ

pred

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Character Theory

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Character Theory

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Character Theory

“A” “a” “ ” ““““”” “”””””



succ *pred* **if then else fi**

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Character Theory

“A”

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succ

pred

if then else fi



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Character Theory

“A” “a” “ ” ““““”” “”””””

succ *pred* **if then else fi**



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Character Theory

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