QUESTION 1. [15 MARKS]

Recall the base -2 representation of an integer, where $(b_n \cdots b_0)_{-2}$ represents $\sum_{i=0}^{n} b_i(-2)^i$, and the $b_i \in \{0,1\}$.

1. Which integers (in our usual, base 10 representation) do the following represent? No justification required.

   (a) $(11111)_{-2} = (11)_{10}$

   (b) $(101010)_{-2} = (-42)_{10}$

   (c) $(10101)_{-2} = (21)_{10}$

   (d) $(111111)_{-2} = (-21)_{10}$

2. What is the base -2 representation of the following integers? No justification required.

   (a) $24 = (1101000)_{-2}$

   (b) $-15 = (110001)_{-2}$

   (c) $30 = (1100010)_{-2}$

Question 2. [15 marks]

Let $p(e)$, $q(d)$, $r(x,d)$, and $s(x,e)$ be unknown predicates, and let K be an unknown domain. Consider statement S1:

$$\text{S1}: \quad \forall e \in K, p(e) \Rightarrow (\exists d \in K, q(d) \wedge (\forall x \in K, r(x,d) \Rightarrow s(x,d))).$$

1. Write a structured proof outline for S1, filling in "$\vdots$" for the missing parts.

Sample solution:

Let $e \in K$

Suppose $p(e)$

Let $d_e = \vdots$ (something depending on e).

Then $d_e \in K$

Also $q(d_e)$.

Let $x \in K$

Suppose $r(x, d_e)$

$\vdots$ (proof that $s(x, d_e)$)

Hence $s(x, d_e)$

Since x is an arbitrary real number, $\forall x \in K, r(x, d_e) \Rightarrow s(x, d_e)$

Since $d_e \in K$, $\exists d_e \in K, q(d) \wedge (\forall x \in K, r(x,d) \Rightarrow s(x, d_e))$

Hence $p(e) \Rightarrow (\exists d \in K, q(d) \wedge (\forall x \in K, r(x,d) \Rightarrow s(x,d)))$

Since e is an arbitrary element of K, $\forall e \in K, p(e) \Rightarrow (\exists d \in K, q(d) \wedge (\forall x \in K, r(x,d) \Rightarrow s(x,d)))$

2. State the negation of S1 in precise notation, moving the negation symbol "$\neg$" as close as possible to the predicates p, q, r, or s.

Sample solution:

$$\exists e \in K, p(e) \wedge (\forall d \in K, \neg q(d) \vee (\exists x \in K, r(x,d) \wedge \neg s(x,d)))$$

QUESTION 3. [15 MARKS]

Let $P = \{g : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$ (the set of functions from the natural numbers to the non-negative real numbers). Let $O(f) = \{g \in P | \exists c \in \mathbb{N}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)\}$. Prove or disprove the following:

$$\forall f \in P, \forall f' \in P, \forall g \in P, (f \in O(g) \wedge f' \in O(g)) \Rightarrow (f + f') \in O(g)$$

SAMPLE SOLUTION: The statement is true.

Let $f \in P$. Let $f' \in P$. Let $g \in P$. Assume $f \in O(g) \wedge f' \in O(g)$.

> Then $f \in O(g)$. (By assumption)
>
> So $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)$. (definition of $f \in O(g)$).
>
> Let $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_1 \Rightarrow f(n) \leq c_1 g(n)$.
>
> Then $f' \in O(g)$. (By assumption).
>
> So $\exists c \in \mathbb{R}, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f'(n) \leq cg(n)$. (definition of $f' \in O(g)$).
>
> Let $c_2 \in \mathbb{R}^+, B_2 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_2 \Rightarrow f'(n) \leq c_2 g(n)$.
>
> Let $c' = c_1 + c_2$ and $B' = \max\{B_1, B_2\}$.
>
>> Then $c' \in \mathbb{R}^+$. (since the positive real numbers are closed under addition).
>> Then $B' \in \mathbb{N}$. (the maximum of two natural numbers is a natural number).
>> So $\forall n \in \mathbb{N}, n' \geq B \Rightarrow n \geq B_1 \wedge n \geq B_2$. (since B' is the maximum of B_1 and B_2).
>> So $\forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) \leq c_1 g(n)$. (since $n \geq B'$, by construction of c_1, and assumption that $f \in O(g)$).
>> So $\forall n \in \mathbb{N}, n \geq B' \Rightarrow f'(n) \leq c_2 g(n)$. (since $n \geq B'$, by construction of c_2, and assumption that $f' \in O(g)$).
>> So $\forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) + f'(n) \leq (c_1 + c_2)g(n)$. (adding the last two inequalities).
>> Hence $\forall n \in \mathbb{N}, n \geq B' \Rightarrow (f(n) + f'(n)) \leq c'g(n)$. (By construction of c').
>
> Since $c' \in \mathbb{R}^+$ and $B' \in \mathbb{N}$, $\exists c' \in \mathbb{R}^+, \exists B' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B' \Rightarrow (f(n) + f'(n)) \leq c'g(n)$.
>
> So $(f + f') \in O(g)$. (by definition).

So $f \in O(g) \wedge f' \in O(g) \Rightarrow (f + f') \in O(g)$.

Since f, f', and g are arbitrary functions in P,
$\forall f \in P, \forall f' \in P, \forall g \in P, (f \in O(g) \wedge f' \in O(g)) \Rightarrow (f + f') \in O(g)$.

Total Marks = 45