Computer Science 236F
St. George Campus

October 14, 2010
University of Toronto

Homework Assignment #2
Due: November 4, 2010 at 6pm

On the cover page of your assignment, you must write ***and sign*** *the following statement: "I have read and understood the homework collaboration policy described in the Course Information handout." Without such a signed statement your homework will not be marked.*

1. (20 marks) In this question you should prove your assertions using only the logical equivalences of Section 5.6 (i.e. without using truth tables).

   (a) (5 marks) Prove that $x \wedge (\neg y \leftrightarrow z)$ is logically equivalent to $((x \rightarrow y) \vee \neg z) \rightarrow (x \wedge \neg(y \rightarrow z))$

   (b) (5 marks) Find a CNF formula that is logically equivalent to $\neg y \rightarrow (x \leftrightarrow z)$. Prove your assertion.

   (c) (10 marks) Classify each formula below as either a tautology, a contradiction or a contingency (satisfiable but not a tautology). Justify your answers.

      i. $((x \vee y) \rightarrow z) \vee (z \rightarrow (x \vee y))$
      ii. $(x \rightarrow y) \wedge (\neg x \rightarrow y)$
      iii. $(x \rightarrow y) \vee (x \rightarrow \neg y)$

Soln

(a)

$x \wedge (\neg y \leftrightarrow z)$	LEQV $x \wedge ((\neg y \wedge z) \vee (y \wedge \neg z))$	($\leftrightarrow$ law)
	LEQV $(x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z)$	(Distr. law)
	LEQV $\neg(x \wedge \neg y \wedge z) \rightarrow (x \wedge y \wedge \neg z)$	($\rightarrow$ law)
	LEQV $(\neg x \vee y \vee \neg z) \rightarrow (x \wedge y \wedge \neg z)$	(De Morgan's law)
	LEQV $((x \rightarrow y) \vee \neg z) \rightarrow (x \wedge y \wedge \neg z)$	($\rightarrow$ law)
	LEQV $((x \rightarrow y) \vee \neg z) \rightarrow (x \wedge \neg(\neg y \vee z))$	(De Morgan's law)
	LEQV $((x \rightarrow y) \vee \neg z) \rightarrow (x \wedge \neg(y \rightarrow z))$	($\rightarrow$ law)

(b)

We will show that the CNF formula $(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z)$ is logically equivalent to $\neg y \rightarrow (x \leftrightarrow z)$.

$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z)$	LEQV $(y \vee (x \vee \neg z)) \wedge (y \vee (\neg x \vee z))$	(Comm. law)
	LEQV $y \vee ((x \vee \neg z) \wedge (\neg x \vee z))$	(Distr. law)
	LEQV $\neg y \rightarrow ((x \vee \neg z) \wedge (\neg x \vee z))$	($\rightarrow$ law)
	LEQV $\neg y \rightarrow (((x \vee \neg z) \wedge \neg x) \vee ((x \vee \neg z) \wedge z))$	(Distr. law)
	LEQV $\neg y \rightarrow ((x \wedge \neg x) \vee (\neg z \wedge \neg x) \vee (x \wedge z) \vee (\neg z \wedge z))$	(Distr. law)
	LEQV $\neg y \rightarrow ((\neg z \wedge \neg x) \vee (x \wedge z))$	(Identity law)
	LEQV $\neg y \rightarrow (x \leftrightarrow z)$	($\leftrightarrow$ law)

(c) You can justify these with truth tables, or:

1.

$((x \vee y) \rightarrow z) \vee (z \rightarrow (x \vee y))$	LEQV $(\neg(x \vee y) \vee z) \vee (\neg z \vee (x \vee y))$	($\rightarrow$ law)
	LEQV $(\neg(x \vee y) \vee (x \vee y)) \vee (z \vee \neg z)$	(Comm. law)

Therefore the formula is logically equivalent to a disjunction of 2 tautologies (namely the tautologies $\neg(x \vee y) \vee (x \vee y)$ and $z \vee \neg z$. Hence it is itself a tautology.

2. This formula is a contingency because the assignment $(x, y) = (1, 1)$ satisfies the formula whereas the assignment $(x, y) = (1, 0)$ falsifies it.

3. $$\begin{array}{rll} (x \to y) \vee (x \to \neg y) & \text{LEQV } (\neg x \vee y) \vee (\neg x \vee \neg y)) & (\to \text{ law}) \\ & \text{LEQV } \neg x \vee (y \vee \neg y) & (\text{Idempotency law}) \end{array}$$

   Since $(y \vee \neg y)$ is a tautology, so is our formula.

2. (20 marks)

   (a) (10 marks) Prove that $\{\neg, \to\}$ is a complete set of connectives.

   (b) (10 marks) Prove that $\{\oplus, \vee\}$ is not a complete set of connectives.

Soln

(a) $\{\neg, \&\}$ is a complete set of connectives, so we only need to show that it is possible to implement & using $\neg$ and $\to$:

$$\begin{array}{rl} \neg(p \to \neg q) & \\ \text{LEQV } \neg(\neg p \vee \neg q) & \to law \\ \text{LEQV } \neg\neg p \& \neg\neg q & DeMorgan's\ law \\ \text{LEQV } p \& q & Double \neg \times 2. \end{array}$$

(b) We define the set G of propositional formulas that use only connectives in $\{\oplus, \vee\}$. G is the smallest set such that:

BASIS: Any propositional variable is in G.
INDUCTION STEP: If $P_1, P_2 \in G$ then so are $P_1 \oplus P_2, P_1 \vee P_2$.

Consider the truth assignment τ that assigns false to all the propositional variables. That is, $\tau(x) = 0$ for every propositional variable x. For $P \in G$, let $S(P)$ be the predicate:

$S(P)$: τ falsifies P.

We use structural induction to prove that $S(P)$ holds for every $P \in G$.

BASIS: P is a propositional variable. Then clearly $S(P)$ holds.

INDUCTION STEP: Let $P_1, P_2 \in G$ for which $S(P_1), S(P_2)$ hold, i.e., τ falsifies both P_1 and P_2. We must show that $S(P)$ holds for any formula $P \in G$ that can be constructed out of P_1 and P_2. There are 2 cases to consider:

CASE 1: P is of the form $P_1 \oplus P_2$. Since τ falsifies P_1 and P_2 it will also falsify P by the truth table of $\oplus$. Hence $S(P)$ holds.

CASE 2: P is of the form $P_1 \vee P_2$. Again, τ falsifies P since both P_1 and P_2 are falsified.

Thus we have proved that any formula that belongs to G is falsified by τ. Now consider the formula $\neg x$ where x is an arbitrary propositional variable. This formula is clearly satisfied by τ. Hence it cannot belong to G, which means that $\{\oplus, \vee\}$ is not a complete set of connectives.

3. (10 marks)

Consider truth assignments containing only the propositional variables x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3, y_4. Every such truth assignment gives a value of 1 (representing true) or 0 (representing false) to each variable. Therefore we can think of a truth assignment τ as determining a 4-bit integer x_τ, where the most significant bit is x_3 and the least significant bit is x_0. In particular, $x_\tau = \tau(x_0) + 2\tau(x_1) + 4\tau(x_2) + 8\tau(x_3)$. Similarly the assignment τ determines a 5-bit integer $y_\tau = \tau(y_0) + 2\tau(y_1) + 4\tau(y_2) + 8\tau(y_3) + 16\tau(y_4)$.

Write a formula that is satisfied by exactly those truth assignments τ for which $y_\tau = x_\tau + 1$. You may use any of the Boolean connectives discussed in the notes. Justify your answer.

Hint: You need to express as a formula the condition under which the 5-bit number $y_4y_3y_2y_1y_0$ is one more than the 4-bit number $x_3x_2x_1x_0$. Think of what this formula would have to say about $y_0, ..., y_4$ first in case $x_0 = 0$, then in case $x_0 = 1$ but $x_1 = 0$ and so on and so forth.

Soln A formula that is satisfied whenever $y_\tau = x_\tau + 1$ is the following:

$$\begin{array}{rcl}
\neg x_0 & \rightarrow & (y_0 \wedge (y_1 \leftrightarrow x_1) \wedge (y_2 \leftrightarrow x_2) \wedge (y_3 \leftrightarrow x_3) \wedge \neg y_4) \\
 & \bigwedge & \\
(x_0 \wedge \neg x_1) & \rightarrow & (\neg y_0 \wedge y_1 \wedge (y_2 \leftrightarrow x_2) \wedge (y_3 \leftrightarrow x_3) \wedge \neg y_4) \\
 & \bigwedge & \\
(x_0 \wedge x_1 \wedge \neg x_2) & \rightarrow & (\neg y_0 \wedge \neg y_1 \wedge y_2 \wedge (y_3 \leftrightarrow x_3) \wedge \neg y_4) \\
 & \bigwedge & \\
(x_0 \wedge x_1 \wedge x_2 \wedge \neg x_3) & \rightarrow & (\neg y_0 \wedge \neg y_1 \wedge \neg y_2 \wedge y_3 \wedge \neg y_4) \\
 & \bigwedge & \\
(x_0 \wedge x_1 \wedge x_2 \wedge x_3) & \rightarrow & (\neg y_0 \wedge \neg y_1 \wedge \neg y_2 \wedge \neg y_3 \wedge y_4)
\end{array}$$

The formula is a conjunction of 5 subformulas. The first such subformula expresses the fact that when $x_0 = 0$ then y_0 should be 1, y_4 should be 0 and all other bits of y should agree with the corresponding bits of x. This ensures that in this case $y_\tau = x_\tau + 1$. The second subformula expresses the conditions that should be true for the bits of y in the case that $x_0 = 1$ and $x_1 = 0$. In particular, y_1 should be 1, y_0 and y_4 should be 0 and all other bits should agree with the corresponding bits of x. In a similar fashion, all the other subformulas take care of the cases when x_0 and x_1 are 1 but $x_2 = 0$, or $x_0 = x_1 = x_2 = 1$ and $x_3 = 0$ as well as the case when all bits of x are 1. Note that all possible settings for the bits of x are covered by these 5 cases.

4. (12 marks) Consider the first order language of arithmetic described in Section 6.2 and the structures $\mathcal{N}$ and $\mathcal{Z}$ described on page 152 in the notes. For each sentence below state whether it is true in $\mathcal{N}$, $\mathcal{Z}$, both or neither. Justify your answer by translating each sentence into a statement in precise English about numbers and then explain why that statement is true or false for the natural numbers or for the integers.

   (a) $\forall x\, \exists y\, L(y, x)$

   (b) $\forall x\, \forall y\, ((L(x, 0) \wedge P(x, x, y)) \rightarrow L(0, y))$

   (c) $\forall x\, \exists y\, \forall z\, \neg P(y, z, x)$

   (d) $\exists x\, \forall y (\neg \approx (0, y) \rightarrow L(x, y))$

Soln

(a) The formula says that for any x there exists a y such that $y < x$. This is clearly true if x, y are integers. However it is not true if they are natural numbers because for $x = 0$ there is no natural number that is less than 0. Hence the formula is true in $\mathcal{Z}$ but false in $\mathcal{N}$.

(b) The formula says that for any x and any y, if $x < 0$ and $y = x^2$, then y is positive. If x, y are integers this is true because if y is the square of a negative number, y is positive. If x, y are natural numbers, then

$x < 0$ is false hence the condition $x < 0$ and $x^2 = y$ is always false, which means that the implication is true. Therefore the sentence is true in both $\mathcal{Z}$ and $\mathcal{N}$.

(c) This says that for any x we can find a y such that for any z, x is not equal to yz. For $x = z = 0$, this cannot be true. Hence the sentence is false in both $\mathcal{Z}$ and $\mathcal{N}$.

(d) The formula says that there exists a number x that is smaller than any nonzero number. If the domain is the integers, this is false as there is no minimum element. If the domain is $\mathbb{N}$ however this is true since any nonzero natural number is greater than 0.

5. (4 marks) Find an equivalent formula F' to

$$F : \exists x R(x,y) \to \exists x(P(x) \vee \neg(\exists y)Q(x,y))$$

such that F' is in PRENEX form and F' has the same free variables as F.

Soln:

$$\begin{array}{rll}
 & \exists x R(x,y) \to \exists x(P(x) \vee \neg(\exists y)Q(x,y)) & \\
\text{LEQV} & \exists x R(x,y) \to \exists x(P(x) \vee \neg(\exists y)Q(z,y)) & \textit{Rename } x \textit{ to } z \\
\text{LEQV} & \exists x R(x,y) \to \exists z(P(z) \vee \neg(\exists u)Q(z,u)) & \textit{Rename } y \textit{ to } u \\
\text{LEQV} & \exists x R(x,y) \to \exists z(P(z) \vee \forall u \neg Q(z,u)) & \textit{Quantifier duality} \\
\text{LEQV} & \forall x(R(x,y) \to \exists z(P(z) \vee \forall u \neg Q(z,u))) & \textit{Quantifier factoring} \\
\text{LEQV} & \forall x \exists z(R(x,y) \to P(z) \vee \forall u \neg Q(z,u)) & \textit{Quantifier factoring} \\
\text{LEQV} & \forall x \exists z \forall u(R(x,y) \to P(z) \vee \neg Q(z,u)) & \textit{Quantifier factoring}
\end{array}$$

It is also possible to derive:

$$\exists z \forall u \forall x(R(x,y) \to (P(z) \vee \neg Q(z,u)))$$

6. (16 marks) For each of the following formulas determine whether it is valid, satisfiable or unsatisfiable. Justify your answer either with a formal proof using logical equivalences and/or by defining appropriate structures.

   (a) $(\forall x Q(x) \to \neg\exists y \exists z R(y,z)) \leftrightarrow \exists x \forall y \forall z(Q(x) \to \neg R(y,z))$

   Soln:

$$\begin{array}{rll}
 & (\forall x Q(x) \to \neg\exists y \exists z R(y,z)) & \\
\text{LEQV} & (\forall x Q(x) \to \forall y \neg\exists z R(y,z)) & \textit{Duality of Quantifiers} \\
\text{LEQV} & (\forall x Q(x) \to \forall y \forall z \neg R(y,z)) & \textit{Duality of Quantifiers} \\
\text{LEQV} & (\exists x(Q(x) \to \forall y \forall z \neg R(y,z)) & \textit{Factoring of Quantifiers} \\
\text{LEQV} & (\exists x \forall y \forall z(Q(x) \to \neg R(y,z)) & \textit{Factoring of Quantifiers} \times 2
\end{array}$$

Therefore, since the left hand side is logically equivalent to the right hand side, the formula is valid.

(b) $(\forall x P(x) \to \exists y \neg Q(y)) \to (\forall x Q(x) \to \exists y \neg P(y))$

Soln:

$$
\begin{aligned}
& \forall x P(x) \to \exists y \neg Q(y) & \\
\text{LEQV } & \neg \forall x P(x) \lor \exists y \neg Q(y) & \quad Arrow\ law \\
\text{LEQV } & \exists y \neg Q(y) \lor \neg \forall x P(x) & \quad commutativity\ of\ \lor \\
\text{LEQV } & \neg \exists y \neg Q(y) \to \neg \forall x P(x) & \quad Arrow\ law \\
\text{LEQV } & \forall y \neg\neg Q(y) \to \exists x \neg P(x) & \quad Duality\ of\ Quantifiers\ \times 2 \\
\text{LEQV } & \forall y Q(y) \to \exists x \neg P(x) & \quad Double\ \neg \\
\text{LEQV } & \forall y Q(y) \to \exists y \neg P(y) & \quad Rename\ x\ to\ y \\
\text{LEQV } & \forall x Q(x) \to \exists y \neg P(y) & \quad Rename\ y\ to\ x
\end{aligned}
$$

Therefore, since the left hand side is logically equivalent to the right hand side, the left hand side logically implies the right hand side, so the formula is valid.

(c) $(\forall x P(x) \to \neg \forall x Q(x)) \leftrightarrow \exists x \forall y (P(x) \to \neg Q(y))$

Soln:

This is satisfiable. Consider a simplification of both sides.

We can say that:

$$
\begin{aligned}
& \forall x P(x) \to \neg \forall x Q(x) & & \\
\text{LEQV } & \neg \forall x P(x) \lor \neg \forall y Q(y) & \quad \to & \quad law \\
\text{LEQV } & \exists x \neg P(x) \lor \exists y \neg Q(y) & \quad Duality\ of & \quad Quantifiers
\end{aligned}
$$

and for the RHS we notice that:

$$
\begin{aligned}
& \exists x \forall y (P(x) \to \neg \mathbf{Q}(\mathbf{y})) & & \\
\text{LEQV } & \exists x \forall y (\neg P(x) \lor \neg Q(y)) & \quad \to & \quad law \\
\text{LEQV } & \exists x \neg P(x) \lor \forall y \neg Q(y) & \quad Factoring & \quad quantifiers
\end{aligned}
$$

To show this is satisfiable, we need to find two structures, one that satisfies both, and one that satisfies one but not the other.

Let S_1 be a structure with domain $D^{S_1} = \{a, b\}$ and define $P^{S_1} = \{a, b\}$ and $Q^{S_1} = \{a, b\}$. Both sides are always false, so the bi-implication is always true.

Now let S_2 be a structure with domain $D^{S_2} = \{a, b\}$ and define $P^{S_2} = \{a, b\}$ and $Q^{S_2} = \{a\}$. The left hand side is always true under S_2, but the right hand side is always false.

(d) $(\forall x (P(x) \to \exists y \neg Q(y))) \to \neg(\forall x Q(x) \to \exists y \neg P(y))$

Soln

Note that:

$$
\begin{array}{rl}
(\forall x(P(x) \rightarrow \exists y \neg Q(y))) & \\
\text{LEQV } \forall x(\neg P(x) \vee \exists y \neg Q(y)) & \rightarrow law \\
\text{LEQV } (\forall x \neg P(x)) \vee \exists y \neg Q(y) & Quantifier\ factoring \\
\text{LEQV } \neg(\exists x P(x)) \vee \neg \forall y Q(y) & Quantifier\ duality \times 2 \\
\text{LEQV } \neg(\exists x P(x) \& \forall y Q(y)) & DeMorgan's\ law
\end{array}
$$

but:

$$
\begin{array}{rl}
\neg(\forall x Q(x) \rightarrow \exists y \neg P(y)) & \\
\text{LEQV } \neg(\neg \forall x Q(x) \vee \exists y \neg P(y)) & \rightarrow law \\
\text{LEQV } \neg(\neg \forall x Q(x) \vee \neg \forall y P(y)) & Quantifier\ duality \\
\text{LEQV } \neg\neg(\forall x Q(x) \& \forall y P(y)) & DeMorgan's\ law \\
\text{LEQV } \forall x Q(x) \& \forall y P(y) & Double\ \neg \\
\text{LEQV } \forall y Q(y) \& \forall y P(y) & Rename\ x\ to\ y \\
\text{LEQV } \forall y Q(y) \& \forall x P(x) & Rename\ y\ to\ x
\end{array}
$$

When the left hand side is true, then either there is a y such that $\neg Q(y)$, in which case the right hand side is false, or there is no x such that $P(x)$ in which case the right hand side is false. So the implication is unsatisfiable.