Computer Science 236 Fall
Evening Section

Sept. 23, 2010
University of Toronto

Assignment 1
Due Thursday, 14th October, 2010 at **6:10pm in tutorial**

On the cover page of your assignment, you must write and sign the following statement: "*I have read and understood the policy on collaboration on homework stated on the course web page.*" Without this signed statement your homework will not be marked.

1. (5 marks) Prove by induction that $\forall n \geq 0, n \in \mathbb{N}$

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

2. (10 marks) Let $T(n,m) = T(n,m-1) + T(n-1,m)$ and $T(s,2) = T(2,s) = s$ for all $s \in \mathbb{N}$.

   Use induction to prove that $T(n,m) \leq \binom{n+m-1}{n-1}$ for all $n, m \in \mathbb{N}$ such that $n, m \geq 2$.

   [Hint: The trick to this question is coming up with the *correct* induction hypothesis.]

3. (10 marks) Recall the Fibonacci numbers,

$$fib(n) = \begin{cases} fib(n-1) + fib(n-2) & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0. \end{cases}$$

   Prove that the Fibonacci numbers satisfy the following identities

$$\begin{array}{lcl} fib(2n-1) & = & (fib(n))^2 + (fib(n-1))^2 \\ fib(2n) & = & (fib(n))^2 + 2fib(n)fib(n-1) \end{array}$$

   for all natural numbers $n \geq 1$.

4. (10 marks) Consider a chocolate bar of dimensions $2 \times n$. How many different ways are there to split the bar up into 2×1 size pieces? For example, a 2×2 bar has 2 ways, a 2×3 bar has 3 ways, a 2×4 bar has 5 ways, etc.

   Prove your answer using induction for all natural numbers $n > 1$.

5. (10 marks) Let M be the smallest set of real-valued matrices such that

   - $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in M$,
   - if $m_1, m_2 \in M$, then $m_1 \cdot m_2 \in M$, and
   - if $m \in M$, $r \in \mathbb{R}$ and $r \neq 0$, then $m_r \in M$, where m_r is obtained from m by multiplying every entry in the first row of m by r.

Prove inductively that every matrix in M is invertible (HINT: prove that the determinant is not 0).

6. (10 marks)

   Prove that the following program is correct with respect to the following Precondition/Postcondition pair.
   (Definition of **div**: if a, b are integers with $b > 0$, then a div b and a mod b are the unique integers such that $a = (a \text{ div } b)b + a \text{ mod } b$.)
   (HINT: You may use without proof the fact that $y = \lfloor\sqrt{m}\rfloor$ if and only if $y^2 \leq m$ and $(y+1)^2 > m$.)

   Precondition: m is an integer, $m \geq 0$.

   Postcondition: The program returns $\lfloor\sqrt{m}\rfloor$.

```
SQRT(m) {
if m = 0 then
    return 0
else
    x := SQRT(m div 4)
    if (2 * x + 1) * (2 * x + 1) ≤ m then
        return 2 * x + 1
    else
        return 2 * x
    end if
end if }
```

7. (15 marks) Prove that the following program is correct with respect to the following Precondition/Postcondition pair. (Intuitively, the program is testing whether an array is sorted.)

   Precondition: m is an integer, $m \geq 1$. A is an integer array.

   Postcondition: If for all i, j such that $1 \leq i < j \leq m$ it is the case that $A[i] \leq A[j]$, then TRUE is returned; otherwise FALSE is returned.

```
k := 1
while k < m and A[k] ≤ A[k + 1] do
    k := k + 1
end while
if k = m then
    return TRUE
else
    return FALSE
end if
```