CSC 2414H (Metric Embeddings) - Take home exam

Due April 27, 2006

General rules : You should solve all questions. Your solution should be your *own* work. You are allowed to use any written material but are not allowed to consult or discuss anything relevant to the course with any other person (except Hamed or Avner for clarifications).

1. Let (X, d) be a metric space with $d(x, y) \in \{1, 2\}$, for every two distinct $x, y \in X$. Moreover for every $x \in X$ we have $|\{z : d(x, z) = 1\}| < B$ for some number B. Show that there is an isometric embedding of X into $\ell_\infty^{O(B \log n)}$. (Hint: Look for a random Frechet embedding.)

2. Let P_n be a path of length n (so the distance between i and j is $|i-j|$). Prove that for every $D > 1$ and $\epsilon > 0$ there exists an $n = n(D, \epsilon)$ such that whenever f embeds P_n into a metric d with distortion at most D, there are $a < b < c$ with $b = \frac{a+c}{2}$ such that f restricted to the subspace $\{a, b, c\}$ of P_n is an embedding with distortion at most $1 + \epsilon$.

3. (a) Let Q_m be the hypercube $\{0, 1\}^m$ equipped with the Hamming distance. Assume $f : Q_m \to \ell_2$ is a nonexapnding embedding that satisifies the following average condition

$$\sum_{a,b \in Q_m} \|f(a) - f(b)\|_2^2 \geq \frac{1}{D} \cdot \sum_{a,b \in Q_m} d_{Q_m}^2(a, b)$$

   Prove that $D = \Omega(m)$.

   (b) Use part (a) to prove that the main structure theorem of ARV is tight (i.e., separation of $1/o(\sqrt{\log n})$ is not possible in some cases).

4. Recall that in non-uniform sparsest cut, we have to find a set $S \subset \{1, \ldots, n\}$ so as to minimize

$$\frac{\sum_{i,j} \gamma_{ij} \delta_S(i,j)}{\sum_{i,j} \eta_{ij} \delta_S(i,j)}$$

where γ_{ij} and η_{ij} are nonnegative numbers. (For the problem of uniform sparsest cut, $\gamma_{ij} = 1$ if $ij \in E$, and 0 otherwise, whereas $\eta_{ij} = 1$ always.)

The following is an SDP relaxation to the problem (verify for youself).

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} \gamma_{ij} \|v_i - v_j\|_2^2 \\ \text{subject to} \quad & \sum_{i,j} \eta_{ij} \|v_i - v_j\|_2^2 = 1 \\ & \|v_i - v_j\|_2^2 \le \|v_i - v_k\|_2^2 + \|v_k - v_j\|_2^2 \qquad \forall i, j, k. \end{aligned}$$

Suppose that the edit distance metric on $\{0,1\}^m$ is embeddable with distortion α_m into an ℓ_2^2 metric. Prove that under this assumption there is an instance of non-uniform sparsest cut so that the above SDP has integrality gap (namely the ratio of the solution to the original problem and the solution to the SDP) at least $\Omega(\frac{\log \log n}{\alpha_{\log n}})$.

5. Consider an β-Liptschitz[1] function $f : S^{n-1} \to \mathbb{R}^+$.

Use Levy's lemma to prove that for $m < \min\{n, \frac{\epsilon^2 n}{(2 \ln 2)\beta^2} - 2\}$, there exists an isometric embedding $g : E_m \to S^{n-1}$ such that $|f(g(x)) - M_f| \le \epsilon$ for every $x \in E_m$. Here, E_m is $\{-1/\sqrt{m}, 1/\sqrt{m}\}^m$ equipped with the ℓ_2 norm, and M_f is the median of f.

[1] i.e. $\forall x, y, |f(x) - f(y)| \le \beta \|x - y\|_2$.