CSC 2414H (Metric Embeddings) - Assignment 3

Due March 22, 2006

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade.

1. For a constant $r \geq 1$, call a metric space (X, d) on n vectors Euc(r) if the distance between $x, y \in X$ is defined as $d(x, y) = \|x - y\|_2^r$. Note that $Euc(1)$ is the set of ℓ_2 metrics.

   (a) Show that any Euc(r) metrics embeds isometrically into a Euc($2r$) metric. (Hint: First show that an n point ℓ_1 metric is Euc(2).)

   (b) Show that constant degree regular expanders require a distortion of at least $\Omega(\log n)$ (the Ω may hida a constant that depends on r) to be embedded into a Euc(r) metric (Hint: use the same Poincaré inequality that we used for ℓ_2, i.e.

   $$n \sum_{ij \in E} |x_i - x_j|^2 \geq \lambda_2 \sum_{i,j} |x_i - x_j|^2,$$

   where λ_2 is the second eigenvalue of the Laplacian.)

2. (a) Consider a metric space (X, d). For every $x \in X$ assign a vector $v_x \in \mathbb{R}^n$, such that $\sum_{x \in X} v_x = 0$. Show that X requires the distortion of at least

   $$\sqrt{\frac{\sum_{\{x,y\}:\langle v_x, v_y \rangle \geq 0} \langle v_x, v_y \rangle d(x,y)^2}{\sum_{\{x,y\}:\langle v_x, v_y \rangle < 0} -\langle v_x, v_y \rangle d(x,y)^2}}$$

   to be embedded into ℓ_2.

   (b) Prove that it is not possible to embed the complete bipartite graph $K_{m,n}$ into ℓ_2 with distortion less than $2\sqrt{1 - \frac{m+n}{2mn}}$.

(c) Give a lowerbound to embedding $K_{m,n}$ into ℓ_1.

3. Let G be a graph with minimum eigenvalue k. Prove that there is an embedding $f : V(G) \to \ell_2$ such that $\|f(i) - f(j)\|_2 = 1$ if $ij \in E$ and $\|f(i) - f(j)\|_2 \geq \sqrt{\frac{k}{k-1}}$ if $ij \notin E$. (Hint: first construct a positive semidefinite matrix and then use that to find an embedding.)

4. (a) Prove that a metric space (X, d) is ℓ_2 if and only if the matrix $M = (m_{ij})$ defined by $m_{ij} = d(i, n)^2 + d(j, n)^2 - d(i, j)^2$ is positive semidefinite.

   (b) Using the above statement (or directly) prove that it is possible to embed every metric space (X, d) into ℓ_2, preserving the order of all $\binom{n}{2}$ distances.