CSC 2414H (Metric Embeddings) - Assignment 2

Due Feb 27, 2006

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade. **special note:** This assignment contains many questions, and so solving 5 questions will already guarantee full mark (but more worth more); notice that question 7 is particularly challenging.

1. Consider the algorithm given in the lecture notes for solving Metrical Task Systems using embeddings into distribution of trees (Theorem 2.1, Lecture note 3). Let us remove the dominance condition from the definition of α-probabilistic embeddings (Definition 1.2 (1), Lecture note 3), i.e. now the assumption on the embedding is

   $$\forall i, j \quad d(i,j) \leq \mathbf{E}_{\tau \in \mathcal{T}}[d_\tau(i,j)] \leq \alpha \cdot d(i,j).$$

   By giving an example show that the randomized algorithm given in the proof of Theorem 2.1 does not guarantee the performance ratio of $\alpha\beta$ (Define a problem by giving a metric space X and specifying a proper cost function $\mathrm{cost}(q,u)$ for requests q and states $u \in X$).

2. Consider a set S consist of n points on a line, and the following procedure for partitioning them with parameter δ: For every point $x \in S$, let $B(x,\delta)$ be the interval of length δ centered at x. Let $B_\delta = \bigcup_{x \in S} B(x,\delta)$. Start picking points $a_1, a_2, \ldots$ from B_δ uniformly at random[1] one at a time. For every i let S_{a_i} be the new points that are covered by $B(a_i,\delta)$, i.e.

   $$S_{a_i} = S \cap \left(B(a_i,\delta) \setminus \bigcup_{j<i} B(a_j,\delta) \right).$$

   The nonempty S_{a_i}'s form the partition.

[1] These points are not necessarily in S.

(a) What is the probability that this partition splits two points x and y with $d := d(x,y)$.

(b) Use the above partitioning in a similar way to the proof of Theorem 3.1 (in Lecture note 3) to prove that S is $O(\alpha)$-probabilistically embaddable into a distribution of 2-HST's.

3. Suppose $p \geq 1$. Show that if (X,d) embeds isometrically into ℓ_p, then it also embeds isometrically into ℓ_p^M, where $M = \binom{n}{2}$, and $n = |X|$.

4. Find an example of an n point metric space $X \subset \mathbb{R}^2$ such that any embedding of X into $\mathbb{R}$ requires a distortion of $\Omega(n)$. (Hint: show you can embed the graph metric of C_n into ℓ_2 with constant distortion. In addition to this fact you may also want to use theorems from previous lecture notes.)

5. Show that it is not possible to embed the n-dimensional hamming cube (i.e. $\{0,1\}^n$ equipped with the ℓ_1 norm) into ℓ_1^k, with constant distortion and $k = o(\log n)$.

6. We are interested in dimension reductions that maintain the area of triangles. For $x, y, z \in \mathbb{R}^k$ denote by (x,y,z) the triangle with vertices x, y, z. Consider a metric space $X \subset \mathbb{R}^n$ and a dimension reduction $f : X \to \mathbb{R}^m$ with distortion $1 + \epsilon$.

(a) We want to show that f does not guarantee any constant upper bound on the distortion of the areas of the triangles: Show that for every $C > 1$ there is an example such that

$$C < \left(\max \frac{\text{area}(x_i, x_j, x_t)}{\text{area}(f(x_i), f(x_j), f(x_t))} \right) \times \left(\max \frac{\text{area}(f(x_i), f(x_j), f(x_t))}{\text{area}(x_i, x_j, x_t)} \right),$$

where $\text{area}(x_i, x_j, x_t)$ is the area of a triangle with edges of length $d(x_i, x_j), d(x_j, x_t)$ and $d(x_i, x_t)$.

(b) Suppose that f guarantees

$$A := \left(\max \frac{\text{area}(x_i, x_j, x_t)}{\text{area}(f(x_i), f(x_j), f(x_t))} \right) \times \left(\max \frac{\text{area}(f(x_i), f(x_j), f(x_t))}{\text{area}(x_i, x_j, x_t)} \right) < (1+\epsilon)^2.$$

Find a constant upper-bound for the distortion of the angles

$$B := \left(\max \frac{\angle(x_i, x_j, x_t)}{\angle(f(x_i), f(x_j), f(x_t))} \right) \times \left(\max \frac{\angle(f(x_i), f(x_j), f(x_t))}{\angle(x_i, x_j, x_t)} \right),$$

where $\angle(x_i, x_j, x_t)$ is defined according to triangle with edges of length $d(x_i, x_j), d(x_j, x_t)$ and $d(x_i, x_t)$.

(c) Is the reverse true? More precisely does $B < (1+\epsilon)^2$ imply $A \leq c$ for some constant $c > 0$?

7. In this question you are going to prove a classical result in the theory of Banach spaces called "Dvoretzky's theorem". The theorem says that an arbitrary norm on $\mathbb{R}^n$ is very similar to Euclidean norm when restricted to a certain subspace of high dimension.

We will need to use Levy's lemma. Denote by $|A|$ the measure of a set $A \subseteq S^{n+1}$. Note that $|S^{n+1}| = 1$.

Lemma 1 (Levy's lemma) *Let $f : S^{n+1} \to \mathbb{R}$ be continuous and let $A = \{x : f(x) = M_f\}$, where M_f is the median*[2] *of f. Then*

$$|A_\epsilon| \geq 1 - \sqrt{\pi/2}e^{-\epsilon^2 n/2},$$

with

$$A_\epsilon := \{t : \rho(t, A) \leq \epsilon\},$$

where ρ is the geodesic distance on the sphere.

Recall that in a metric space (X, d) a θ-net is a set $S \subseteq X$ such that $d(x, S) \leq \theta$ for every $x \in X$.

(a) Prove that for every normed space X of dimension k, there exists a θ-net N in $S(X) = \{x \in X : \|x\| = 1\}$ with

$$|N| \leq (1 + 2/\theta)^k \leq e^{k \log 3/\theta}.$$

(Hint: Use a greedy algorithm to find the θ-net.)

(b) Consider an β-Liptschitz[3] function $f : S^{n+1} \to \mathbb{R}^+$.

Use Part (a) and Levy's lemma (note that an β-Liptschitz function is continuous) to prove that there exists a subspace $E \subseteq \mathbb{R}^{n+2}$ with $\dim(E) = \epsilon^2 n/(2 \log 4/\theta)$, and a θ-net N in $S(E) = S^{n+1} \cap E$ such that

i. $|f(x) - M_f| \leq \epsilon$ for all $x \in N$, and

[2] i.e. $|\{x : f(x) \geq M_f\}| \geq 1/2$ and $|\{x : f(x) \leq M_f\}| \geq= 1/2$.

[3] i.e. $|f(x) - f(y)| \leq \beta\|x - y\|_2$.

ii. $|f(x) - M_f| \leq \epsilon + \beta\theta$ for all $x \in E \cap S^{n+1}$.

(Hint: Similar to the proof of Johnson-Lindestrauss theorem choose E at random)

(c) Consider a normed space $X = (\mathbb{R}^{n+2}, \|\cdot\|)$. Let $f(x) = \|x\|$, and assume that $|f(x) - M_f| \leq \beta\epsilon$ for all x in a θ-net N of $E \cap S^{n+1}$ for some subspace E of $\mathbb{R}^{n+2}$. Prove that

$$\frac{1-2\theta}{1-\theta}M_f - \frac{\beta\epsilon}{1-\theta} \leq \|x\| = f(x) \leq \frac{1}{1-\theta}M_f + \frac{\beta\epsilon}{1-\theta},$$

for all $x \in E \cap S^{n+1}$.

(Hint: Do not use the previous parts; First prove the upper bound; Find $\{y_i\}_{i=1}^{\infty}$ in N and $\{\delta_i\}_{i=1}^{\infty}$ in $\mathbb{R}$ with $|\delta_i| \leq \theta^{i-1}$ such that $x = y_1 + \sum_{i=2}^{\infty} \delta_i y_i$).

(d) Use the previous parts to prove that for any $\delta > 0$ there exists a $c(\delta) > 0$ such that the following holds:

Consider an arbitrary normed space $X = (\mathbb{R}^{n+2}, \|\cdot\|)$, and let $f(x) = \|x\|$. From equivalence of norms in finite dimension, we already know that there exists $\alpha, \beta > 0$ such that

$$\alpha^{-1}\|x\|_2 \leq \|x\| \leq \beta\|x\|_2.$$

There exists a subspace E of $\mathbb{R}^{n+2}$ with $\dim(E) \geq c(\delta)\cdot n\cdot(M_f/\beta)^2$ and

$$(1-\delta)\cdot M_f\cdot\|x\|_2 \leq \|x\| \leq (1+\delta)\cdot M_f\cdot\|x\|_2,$$

for every $x \in E$. (This means that E is similar to a Euclidian space or more precisely the identical embedding of E into ℓ_2 has distortion $\frac{1+\delta}{1-\delta}$.)