# CSC2411 - Linear Programming and Combinatorial Optimization*
# Lecture 6: LP Duality, Complementary slackness, Farkas Lemma, and von Neumann min-max principle.

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**Summary:** In this lecture, we further discuss the duality of LP. We prove duality theorems, discuss the slack complementary, and prove the Farkas Lemma, which are closely related to each other. At last, we discuss an application: von Neumann Min-Max theorem.

## 1 Primal and Dual

Recalling the question 1 from the assignment, at the first step, we formalize it as:

$$\begin{aligned} &\min \quad t \\ s.t. \quad & |ax_i - b - y_i| \le t, \forall i \end{aligned}$$

then we get the following LP as our primal linear programming:

$$\begin{aligned} &\min \quad t \\ s.t. \quad & -ax_i + b + t \ge -y_i, \forall i \\ & ax_i - b + t \ge y_i, \forall i \\ & a, b, t \gtrless 0 \qquad \text{these variables are unconstrained} \end{aligned}$$

We use a figure in the following to illustrate the problem.

* Lecture Notes for a course given by Avner Magen, Dept. of Computer Sciecne, University of Toronto.

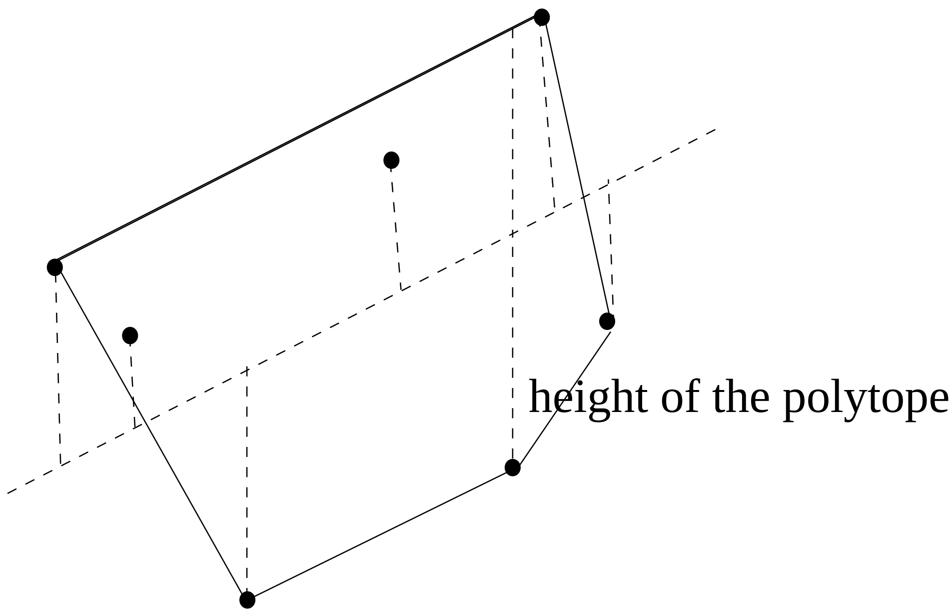


Figure 1: The geometry representation of question 1 in Assignment 1

From our previous lectures and tutorials, we know how to get its dual as following:

$$\begin{aligned}
& \max \quad -\sum \alpha_i y_i + \sum \beta_i y_i \\
s.t. \quad & -\sum \alpha_i x_i + \sum \beta_i x_i = 0, && (1) \text{ this is for variable a} \\
& \sum \alpha_i - \sum \beta_i = 0 && (2) \text{ this is for variable b} \\
& \sum \alpha_i + \sum \beta_i = 1 && (3) \text{ this is for variable t} \\
& \alpha_i, \beta_i \geq 0, \forall i
\end{aligned}$$

Let us define $\gamma_i = 2\alpha_i$ and $\delta_i = 2\beta_i$, then our problem can be transformed to:

$$\begin{aligned}
& \max \frac{1}{2}\left(\sum \delta_i y_i - \sum \gamma_i y_i\right) \\
s.t. \quad & \sum \gamma_i x_i = \sum \delta_i x_i \\
& \sum \gamma_i = \sum \delta_i = 1 \\
& \gamma, \delta \geq 0
\end{aligned}$$

Notice that $\gamma, \delta$ are coefficients of convex combinations. This motivates us to define $p = \sum \gamma_i(x_i, y_i)$ and $q = \sum \delta_i(x_i, y_i)$, then the problem is transformed to:

$$\begin{aligned} & \max q_y - p_y \\ s.t. \quad & q_x = p_x \\ & p, q \in conv(S) \qquad \text{where } S \text{ is the convex hull of } (x_1, y_1), ..., (x_n, y_n) \end{aligned}$$

The meaning of the above formulation is that $p, q$ range over all possible pairs of points in the convex hull $S$ generated by input points, and we want to find the convex hull's maximal height at some point $x$.

It is easy to find that $\min t$ can not be smaller than half of the height found in the dual problem. The half of the height here is a bound for the original primal problem. It would be interesting to see a simple proof which shows us the above observation. In the following section, we will discuss such proofs.

## 2 Weak and Strong Duality Theorem

From what we have known so far, the following figure gives us a rough idea of the possible values for primal and dual's solutions.

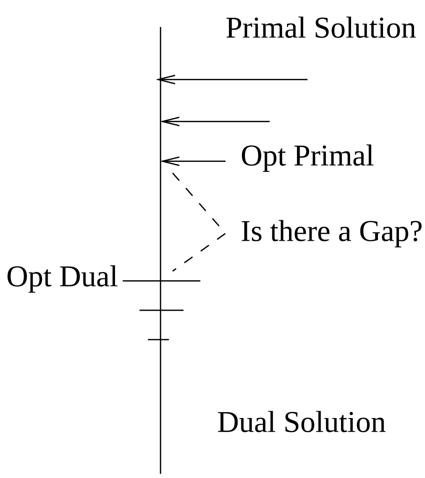


Figure 2: The relationship between primal and dual solutions by assuming both being feasible

This figure is captured by the following theorems. Before we start to prove the theorems, we want to point out that we will prove most of our theorems for linear programming in standard forms. But this does not prevent us from applying them to other forms, since different forms can be defined by each other and the proofs can be extended too.

**Theorem 2.1. Weak Duality Theorem**
*If $x$ is feasible for the primal and $y$ is feasible for the dual, then $(c, x) \geq (y, b)$*

*Proof.* By the construction of the primal and dual, we immediately get

$$(c, x) \geq yAx = (y, b)$$

□

From the above theorem, we do not know whether there is a gap between the optimal primal solution and the optimal dual solution when both problem are feasible, and whether feasibility can be related to optimum. The following strong duality theorem tells us that such gap does not exist:

**Theorem 2.2. Strong Duality Theorem**
*If an LP has an optimal solution then so does its dual, and furthermore, their optimal solutions are equal to each other.*

An interesting aspect of the following proof is its base on simplex algorithm. Particularly, we will utilize the property of simplex algorithm's proof at termination.

*Proof.* Let the primal and dual be:

$$\begin{array}{ll} Primal & Dual \\ \min(c, x) & \max(y, b) \\ Ax = b & yA \leq c \\ x \geq 0 & y \gtreqless 0 \end{array}$$

Assuming that our primal problem has an optimal solution, then at the termination point of simplex algorithm with a basis $B$ and the remaining columns $N$, we have the following inequalities:

$$\bar{c} = c_N - c_B A_B^{-1} A_N \geq 0$$

$$c_B A_B^{-1} A_N \leq c_N.$$

Since $c_B A_B^{-1} A_B = c_B$, we have:

$$c_B A_B^{-1} A \leq c$$

Indeed, let $y = c_B A_B^{-1}$, then $yA \leq c$. Therefore, if the primal has an optimal solution, then we can find a feasible solution $c_B A_B^{-1}$ for the dual. Since $x = A_B^{-1} b$, we can get the following equation:

$$(y, b) = c_B A_B^{-1} b = (c, x)$$

Therefore, we can conclude that if primal has an optimal solution, the dual is feasible and has a solution with the value equal to the primal optimum. □

## 2.1 Primal and Dual's Possible Category

Considering the pair of primal and dual problems, our discussion up to now has informed us three different possible combinations: infeasible, feasible but unbounded, and feasible bounded. Table 1 represents their possible relationships.

| | | Primal | | |
|---|---|---|---|---|
| | | Has Optimum | Unbounded | Infeasible |
| Dual | Has Optimum | $\surd^*$(Same Optimum) | $\times^0$ | $\times^*$ |
| | Unbounded | $\times^0$ | $\times^0$ | $\surd^0$ |
| | Infeasible | $\times^*$ | $\surd^0$ | $\surd$ |

Table 1: "*" represents the conclusion of strong duality, "0" represents the conclusion of week duality corollary

As for the (Infeasible, Infeasible) entry, it is quite easy to see it is possible: suppose the primal is infeasible, then we write its dual and add more constraints to make it infeasible, and the primal obtained from the new infeasible dual still is infeasible.

We observe that the dual of the dual problem is the primal. Therefore the table must be symmetric.

# 3 Complementary Slackness

For the following discussion, we will use the linear programming in canonical forms. We consider the following primal and dual:

$$\begin{array}{ll} \min \quad (c,x) & \qquad \max \quad (y,b) \\ Ax \geq b & \qquad yA \leq c \\ x \geq 0 & \qquad y \geq 0 \end{array}$$

Previously we have emphasized the special roles of the inequalities that holds as equalities for a certain solution, particularly for the optimal solution. In the context of primal and dual problems, the following theorem gives an ultimate expression of this balance for $x, y$ to be respective optima in the primal-dual.

**Theorem 3.1. Complementary Slackness**

*Let $x, y$ be feasible solutions for primal and dual problems respectively, then they are optimal solutions iff:*

$$\begin{aligned} &\forall i, x_i = 0 \quad \textit{or} \quad (yA)_i = c_i \\ &\forall j, y_j = 0 \quad \textit{or} \quad (Ax)_j = b_j \end{aligned}$$

*Proof.* Since $x, y$ are feasible, then,

$$(c, x) \geq yAx \geq (y, b).$$

By strong duality theorem, $x, y$ are optima iff $(c, x) = (y, b)$. Furthermore, $(c, x) = (y, b)$ iff all the above three terms are the same.

What can we get from the above equations? Let us consider the first equation$(c, x) = yAx$, which can be rewritten as

$$(c - yA, x) = 0.$$

Since $x \geq 0$ and $yA \leq c$, we can conclude that $(c, x) = yAx$ iff for all $i$, either $(y, a_i) = c_i$ (whenever $x_i > 0$), or $x_i = 0$.

Similarly, the second equation $yAx \geq (y, b)$ can be rewritten as $(y, Ax - b) = 0$, from which we know that $yAx = (y, b)$ iff either $(a_i, x) = b$(whenever $y_i^* > 0$), or $y_i^* = 0$. □

# 4 Linear System's Feasibility and Farkas Lemma

Now let us discuss the question of whether a set of inequalities and equations is feasible. From our intuition,we know if we multiply the equations by any scalars and add them up, and if we can not get a valid equation, then we know the linear system is infeasible.

Consider the following example:

$$\begin{aligned} &x_1 + 2x_2 = 5 && (1) \\ &x_3 - x_2 = -3 && (2) \\ &x \geq 0 \end{aligned}$$

if we multiply first equation by 1, second equation by 2, and add them up, we get

$$x_1 + 2x_2 + 2x_3 - 2x_2 = 5 - 2 \times 3$$

which is

$$x_1 + 2x_3 = -1.$$

Since $x \geq 0$, we know the linear system is not feasible.

This example demonstrates a simple way to prove the infeasibility of a linear system. To formalize the above method, we write it as: if $\exists y, yA \geq 0$ and $(y, b) < 0$, then $Ax = b, x \geq 0$ is infeasible. Moreover, this can be strengthened to a sufficient and necessary condition. That is, the reverse direction holds too: if a linear system is infeasible, then there must be a ”linear proof” to that. Farkas lemma captures this idea formally.

## 4.1 Farkas Lemma

**Farkas Lemma**

Farkas Lemma was attributed to J. Farkas in 1894. It is useful to think of it as the geometric version of duality theorem.

**Theorem 4.1.** *$Ax = b, x \geq 0$ is feasible,* $\Longleftrightarrow (\forall y, yA \geq 0 \Rightarrow (y, b) \geq 0)$.

*Proof.* The $\Rightarrow$ direction is trivial: let $x$ be a solution to $Ax = b, x \geq 0$. If $yA \geq 0$, then $yAx \geq 0$. Since $yAx = (y, b)$, we have $(y, b) \geq 0$.

Now let us consider the other direction $\Leftarrow$, which is more interesting and complicated. In fact, this is not surprising as it is analogous to strong duality theorem.

For a set of vectors $v_1, v_2, ..., v_n$, we define $cone(v_1, v_2, ..., v_n) = \{\sum_i \lambda_i v_i | \lambda_i \geq 0\}$. Figure 3 illustrates the definition.

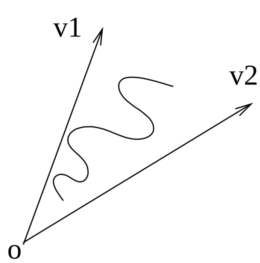


Figure 3: the illustration of the cone’s definition

If we take the columns of $A$ as the vectors, then the infeasibility of $x \geq 0$ with $Ax = b$ is same as saying $b \notin cone(A_1, A_2, ..., A_n)$.

Since $yA \geq 0 \Leftrightarrow \forall i (y, A_i) \geq 0$, we can view $y$ as a hyperplane that go through the origin with all $A_i$ on one side and $b$ on the other side.

Let $K = cone(A_1, A_2, ..., A_n)$, and let $p$ be the closest point to $b$ in $K$. which must exist since the distance from $b$ is a continous function and $K$ is a closed set. We first argue that for every point $z \in K$, $(z - p, b - p) \leq 0$.

If the above claim did not hold, we could find another point $z'$ with smaller distance to $b$ than the distance to $p$. In figure 4, we illustrate the relationships of $b, p, z$ and $z'$.

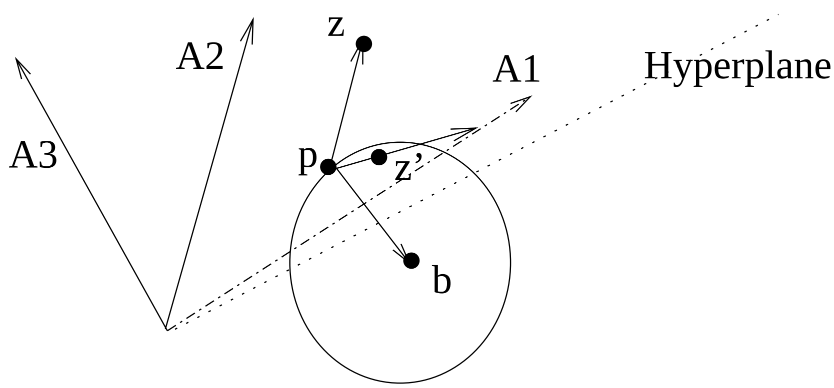


Figure 4: the illustration of the claim and the relationship of z, p, b

Let $Aw = p$, and $y = p - b$. From the above fact, we know that $\forall x \geq 0, (Ax - Aw, y) \geq 0$. In particular, for $x = w + e_i$, we have

$$\forall i, (Ae_i, y) \geq 0 \Rightarrow \forall i, (A_i, y) \geq 0$$

(notice that $w \geq 0$ and $e_i$ is a unit vector with entry $i$ equal to 1, therefore $x \geq 0$.)

As a result, $y$ puts $K$ on one side. Now we need to show that b is put to the other side.

We know that $(p - b, b - p) < 0$. Moreover, since $p$ is the closed point to $b$ in $K$ and $\vec{0} \in K$, from previous discussion, we have $(p - b, p - \vec{0}) \leq 0$. Adding these two terms together, we have:

$$(y, b) = (p - b, b) = (p - b, b - p) + (p - b, p - \vec{0}) < 0,$$

which means $b$ is put to the other side of $y$ than the side of $K$.

Therefore, if $Ax = b, x \geq 0$ is not feasible, we can find a $y$, such that $yA \geq 0, (y, b) < 0$. This finishes the proof. □

# 5 Application: von Neumann min-max principle

A zero sum game is a game with 2 players, in which each player has a finite set of strategies. The payoff to the first player is determined by the strategies chosen by both players, and the payoff to the second player is the negation of the payoff to the first. So the sum of their payoffs is zero. The following Paper-Scissor-Stone game is a zero sum game.

| | Column Player | | | |
|---|---|---|---|---|
| | | Paper | Stone | Scissor |
| Row player | Paper | 0 | 1 | -1 |
| | Stone | -1 | 0 | 1 |
| | Scissor | 1 | -1 | 0 |

Table 2: Paper-Scissor-Stone game's strategy matrix

If the column player plays strategy $i$, and the row player plays strategy $j$, the payoff to the column player is $a_{ij}$. If column player plays first, she will get profit:

$$\max_j \min_i a_{ij}$$

If we reverse the order, she will get profit:

$$\min_i \max_j a_{ij},$$

For the above two terms, we have: $\min_i a_{ij} = -1, \forall j; \max_j a_{ij} = 1, \forall i$. So it is easy to know that there is a big advantage to play second in the above game.

Now what if the player exposes a probability vector that will determine her strategies(i.e. mixed strategies). Since each player's strategy is determined by a probability distribution, the order of playing game becomes less important. Let us define $\Delta_k = \{\alpha \in R^k | \alpha \geq 0, \sum \alpha_i = 1\}$.

For mixed strategy games, we compare the following two objects:

$$\text{Column player play first: } \max_{x \in \triangle_m} \min_{y \in \triangle_n} yAx,$$

$$\text{Column player play second: } \min_{y \in \triangle_n} \max_{x \in \triangle_m} yAx,$$

where $x, y$ are probability vectors and $yAx = \sum x_j y_i a_{ij}$

It is easy to see that $\max_x \min_y yAx \leq \min_y \max_x yAx$. Is it possible that the equality holds? The following theorem give a positive answer

**Theorem 5.1. Von Neumann min-max Theorem(Principle)** *There exist $x, y$ such that*

$$\max_{x \in \triangle_m} \min_{y \in \triangle_n} yAx = \min_{y \in \triangle_n} \max_{x \in \triangle_m} yAx$$