CSC 2411H - Assignment 4

Due Apr 11, 2005

1. (a) Let A be a totally unimodular matrix and let l_1, l_2, u_1 and u_2 be integral vectors. Show that the LP relaxation of the following Integer Program is *exact* (i.e. all its vertices are integral).

$$\begin{array}{ll} & \min\langle x, c\rangle \\ \text{s.t.} & \\ & l_1 \leq Ax \leq u_1 \\ & l_2 \leq x \leq u_2 \end{array} .$$

   (b) Let A be an $m \times n$ integer matrix of rank m. Show that A is unimodular (that is, the determinant of all $m \times m$ submatrices of A is $-1, 0$ or 1) if and only if for every integral vector b the vertices of the polyhedron

$$P = \{x | x \geq 0, Ax = b\}$$

   are all integral.

2. In class we saw that if G is a bipartite (undirected) graph, then the relaxation for the IP for the maximum weight matching is exact.
   (a) Show that if G is not bipartite then the relaxation is not exact.
   (b) Show that the optimal solution of the relaxation is half integral (as in the case of Vertex Cover).

3. (a) Analyze the approximation ratio of the following randomized algorithm for Max-Sat: Each of the variables is set True/False uniformly and independedntly. What can be said if all clauses contain at least k different variables?
   (b) Use the above to improve the $(1 - e^{-1})$-approximarion algorithm for Max-Sat that was presented in class, and to get a (deterministic) $3/4$-approximation algorithm to the problem.

4. Consider the IP for vertex cover problem that was discussed in class. Recall that for the case where G is a bipartite graph the problem is easy as the LP relaxation to the IP is exact.

   (a) Suppose G has a triangle (three vertices x_1, x_2, x_3 so that $x_1x_2, x_2x_3, x_3x_1 \in E(G)$). Then we can add to the IP the constraint $x_1 + x_2 + x_3 \geq 2$. Why is this constraint valid?

   (b) Generalize to an odd cycle C. In other words, write a valid inequality asociated with C that can be added to the IP that still describes the original problem, but which will allow for a tighter relaxation when the integral constraints are removed.

   (c) Suppose we add all constraints for all odd cycles. Let H be a graph on n vertices, that has no odd cycle of size $\leq \log n/10$ and whose minimum vertex cover is of size $n-o(n)$. The existence of such graphs can be shown using probabilistic methods. Consider the IP for H that contains all odd-cycle constraints (plus the usual edge constraints), and show that there is still integrality gap of $2 - o(1)$.

5. Consider Max-Cut with the additional constraint that specified pairs of vertices be on the same/opposite sides of the cut (assume these additional constraints do not lead to inconsistency). Specifically, there are two sets J_+ and J_- of pairs of vertices, so that if $\{x, y\} \in J_+$ then x and y must be at the same side of the cut and if $\{x, y\} \in J_-$ then x and y must be on opposite side of the cut.

   (a) Give a vector program relaxation to this probem.

   (b) Show that Goemans-Williamson algorithm can be adapted to this problem so as to maintain the same approximation factor.

   (c) Back to the original Max-Cut problem. What is the integrality gap for the GW relaxation for C_3,C_4 and (bonus) C_5?