CSC 2411H - Assignment 4

Due April 16, 2009

1. (a) Use the fact (shown in tutorial) that the optimal solution for the Vertex-Cover LP relaxation is half integral (namely it has values from $\{0, 1/2, 1\}$) to show that the integrality gap is always strictly smaller than 2. (Notice, this may still mean that it is smaller from 2 by an amount that is vanishing with n. If you can show that it is a constant smaller than 2, you don't need to solve the rest of the assignment or take the exam to ensure an A^+).

   (b) In class we saw that the matching polytope (the polytope associated with the LP relaxation for maximal mataching) is integral for bipartite graphs. Prove that the vertices of the matching polytope are always half-integral,i.e., points with values from $\{0, 1/2, 1\}$. Hint: think about generalizing the connection with Totally unimodular matrices.

2. Consider the Minimum Set Cover problem from class with the restriction that each element is in at most f sets. Remember that in class we saw an approximation algorithm for this special case based on primal/dual. Give another approximation algorithm for this problem with approximation factor f that is *not* based on the dual.

3. In this question we are going to show that the standard LP for Set-Cover has integrality gap $\frac{\log n}{2} - \Theta(1)$. Consider the instance where the universe

$$U = \{x \in \{0,1\}^n : x \neq \mathbf{0}\},$$

is the set of all strings of length n except all zeros, and for each subset $T \subseteq \{1, \dots, n\}$ we have a set,

$$S_T = \{x \in U : \sum_{i \in T} x_i \text{ is odd.}\}.$$

For example $S_{\{i,j\}}$ is the set of all x which are one in exactly one of the two coordinates i and j.

   (a) Show that any set cover for this instance has size at least n.

   (b) Show that the standard set cover LP has a solution with objective value 2.

   (c) Conclude that the integrality gap is at least $\frac{\log N}{2} - \Theta(1)$ for N vertex graphs.

4. Recall the example from class in which you are given two sets of points in $\mathbb{R}^n$. $\mathcal{P} = \{p_1, \dots, p_r\}$, and $\mathcal{Q} = \{q_1, \dots, q_s\}$. The goal is to to find an ellipse centred at the origin, that includes all $p_i \in \mathcal{P}$ and excludes all $q_i \in \mathcal{Q}$ (where we allow q_i to lie on the boundary). Recall that an ellipse centred at the origin is $\{y | y^t By \leq 1\}$

for some PSD matrix B. The problem can be formulated as the following feasibility SDP program.

$$\begin{array}{c} \forall p_i \in \mathcal{P}\ p_i^t X p_i \le 1 \\ \forall q_i \in \mathcal{Q}\ q_i^t X q_i \ge 1 \\ X \text{ is PSD} \end{array} \tag{1}$$

(a) Write the above program in the form

$$\begin{array}{c} A_i \bullet X \ge b_i, \quad i = 1 \dots m \\ X \succeq 0 \end{array}$$

(recall $A \succeq B$ simply means that $A - B$ is PSD.)

(b) Try to develop "SDP Farkas lemma" for the general program of the above form, that should say that either the program of the format above is feasible or that another program is feasible. You don't need to prove the lemma, just the reason behind the step producing the other program. Repeat the process for the specific problem of the separating ellipsoid.

Remarks: You should use intuition from the usual Farkas Lemma and from duality of SDP (including the fact that A is PSD if and only if $A \bullet B \ge 0$ for all B which is PSD.

5. Find the integrality gap of the Goemans-Williamson algorithm for Max-Cut for the cycles C_4 and C_5.