CSC 2402H - Assignment 2

Due Nov 10, 2009

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade.

1. Show that it is possible that both a primal and dual LP are infeasible.

2. Consider the system that consists of the linear constraints $A\mathbf{x} = b$ (where A is m by n matrix) as well as the set of two-sided inequalities

   $$l_i \leq \mathbf{s_i} \cdot \mathbf{x} \leq u_i$$

   $i = 1, 2 \ldots, r$. ($\mathbf{s_i}$ are some nonzero vectors in $\mathbb{R}^n$). Assume that this system is infeasible. What does Farkas lemma imply in this case? Try to show that only $m + r$ coefficients are needed for a combinations of the l_i, u_i and b_i that produces the proof of infeasibility of the system, as opposed to the "generic" quantity of $m + 2r$.

3. (a) Consider a zero-sum between a row player and column player given by the matrix.

   $$\begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix}$$

   where the entries are the payoffs for the row player.

   Determine the optimal mixed strategies for the game (including optimality proof).

   (b) Consider a zero-sum game of two players with the player I having m strategies and player II has $n \geq m$ strategies. Prove or disprove: Player II has an optimal mixed strategy that uses no more than m of her pure strategies (in other words her strategy vector is supported on no more than m coordinates).

4. Let $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \ldots, q_s$ be points in $\mathbb{R}^n$, and let $P = \text{conv}(p_1, p_2, \ldots, p_r)$ and $Q = \text{conv}(q_1, q_2, \ldots, q_r)$ be their convex hulls (recall that $\text{conv}(s_1, \ldots, s_n)$ is the set $\{\sum_i \lambda_i s_i | \lambda_i \geq 0; \sum_i \lambda_i = 1\}$.)

   We say that a hyperplane H is a strict separating hyperplane between P and Q if H separates P and Q and is disjoint with any of them. Formally, $P \subset H^- \setminus H$ and $Q \subset H^+ \setminus H$, where H^- and H^+ are the two halfspaces associated with H. Show that P and Q have a strict separating hyperplane if and only if they are disjoint.

5. Let $E = \{x : (x - c)^t Q^{-1}(x - c) \leq 1\}$ be an ellipsoid enclosing a (nonempty) polytope P with smallest possible volume. Is it possible that c (the centre of E) does not belong to P? Either give an example of such a polytope and its smallest enclosing ellipsoid or prove that it is impossible.

6. Give an example of a semidefinite programme with integer coefficients that has a unique optimal solution that is irrational.