CSC2402 - Fall 2009
Assignment 4

due on Wednesday, Dec 16th, at 2pm

Problem 1 [15pt] The purpose of this problem is to establish a few facts that you might find useful in other problems. Even if you can not prove one of the following parts you can still use them in other questions.

1. [5pt] Prove that for any $n \geq 2$ unit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ the following is true:

$$\max_{i \neq j} \mathbf{v}_i \cdot \mathbf{v}_j \geq -1/(n-1).$$

2. [5pt] Prove that for any n the above bound is tight. In other words there are vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that any two of them have inner product at most $-1/(n-1)$.

3. [5pt] Consider a symmetric $(n+1) \times (n+1)$ matrix A of the following form,

$$A = \begin{pmatrix} \gamma & b_1 & b_2 & \cdots & b_n \\ b_1 & & & & \\ b_2 & & & X & \\ \vdots & & & & \\ b_n & & & & \end{pmatrix},$$

where X is a symmetric $n \times n$ matrix and $\gamma > 0$ is a positive real. Prove that A is positive semidefinite if and only if $X - \frac{1}{\gamma} bb^t$ is positive semidefinite. Here bb^t, the outer product of b with itself, is defined as

$$(bb^t)_{ij} = b_i b_j.$$

Problem 2 [50pt] Remember that in the Maximum Independent Set problem we are given a graph G and we are interested in the largest set of vertices that has no edges inside it. The following question concerns the canonical SDP relaxation of this problem seen below.

$$\begin{array}{rc} \max & \sum_i \mathbf{v}_0 \cdot \mathbf{v}_i \\ \text{subject to} & \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^{n+1}, \\ & ||\mathbf{v}_0||_2^2 = 1, \\ \forall i & \mathbf{v}_0 \cdot \mathbf{v}_i = \mathbf{v}_i \cdot \mathbf{v}_i, \\ \forall ij \in E(G) & \mathbf{v}_i \cdot \mathbf{v}_j = 0. \end{array}$$

We are going to show that the integrality gap of this relaxation is at least $\Omega(\sqrt{n}/\log n)$. (It takes much more effort but it can be shown that the integrality gap is actually close to n.)

1. [5pt] Prove that the above SDP is a relaxation of the problem. In other words, show that for any subset of the vetices S, there is a set vectors that satisfy the above conditions if and only if S is an independent set, and has objective value $|S|$.

2. [8pt] Remember that the standad LP relaxation of Maximum Indepednent Set has a very big integrality gap when $G = K_n$.

   Show that the SDP relaxation has *no* integrality gap for K_n, namely the objective value is 1 for this graph. (You may want to use the result of question 1.)

3. [10pt] Consider the random graph $G(n, 1/2)$ which is the graph on n vertices where every possible edge occurs independently with probability $1/2$. Show that with probability $1 - o(1)$, the size of the biggest independent set of $G(n, 1/2)$ is less than $3 \log n$.

   Hint: What is the probability that a subset of the vertices $S \subseteq V(G)$ of size $3 \log n$ is an indepedent set?

4. (bonus) Show that with probability $1 - o(1)$ the objective value of the SDP above on $G(n, 1/2)$ is at least $c\sqrt{n}$ for some constant $c > 0$.

   Hint: You have to construct a solution to the above SDP with high objective value. Instead of constructing a vector solution, try constructing a matrix (of the inner product) and showing that it is PSD.

5. [5pt] Consider the following Semidefinite program,

$$\begin{array}{rc} \min & ||\mathbf{u}_0||_2^2 \\ \text{subject to} & \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^{n+1}, \\ \forall i & 2\mathbf{u}_0 \cdot \mathbf{u}_i + \mathbf{u}_i \cdot \mathbf{u}_i = -1, \\ \forall i < j, ij \notin E(G) & \mathbf{u}_i \cdot \mathbf{u}_j = 0. \end{array}$$

   Show that the optimum of this program is always more than or equal to the optimum of the SDP relaxation of Independent Set we saw above. (as a side note, this is a special case of weak duality for Semidefinite Programming.)

   Hint: The following simple fact may be useful: for vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ the following holds,

$$\sum_{ij} (\mathbf{a}_i \cdot \mathbf{a}_j)(\mathbf{b}_i \cdot \mathbf{b}_j) \geq 0.$$

6. (bonus) Show that for $G(n, 1/2)$ with probability $1 - o(1)$ the objective value of the SDP is at most $C\sqrt{n}$ for some constant C.

For the bonus parts (4 and 6) you would most probably want to use the following theorem (for a proof see Z. Furedi and J. Komlos, "The eigenvalues of random symmetric matrices", Combinatorica 1 (1981)). Fix constants ν, σ, and μ and consider a random symmetric n by n matrix $A = (a_{ij})$ where the (upper triangular) entries are chosen independently according to some distribution,[1]

[1] Technically, the theorem also requires that these distributions are bounded but this is not important for this question.

- A is symmetric, that is a_{ij} is always equal to a_{ji},
- For $i \neq j$, $\mathbb{E}[a_{ij}] = \mu$ and a_{ij} has variance σ^2,
- For any i, $\mathbb{E}[a_{ii}] = \nu$.

Then with probability $1 - o(1)$, all the eigenvalues of A are at least $-3\sigma\sqrt{n}$.

Problem 3 [15pt] Remember the KMS algorithm for coloring a 3-colorable graph with maximum degree Δ from class. Remember the algorithm used $\Delta^{\log_3 2}$ colors to color such graphs. Change the algorithm slightly so that it can color k-colorable graphs. What is the number of colors the algorithm will use?

Hint: The algorithm will actually work and will use less colors than the trivial greedy algorithm, however the number of colors used is not a particularly pretty expression.

Problem 4 [20pt] Consider the IP for vertex cover problem that was discussed in class. In class we saw that the "odd-cycle constraints" which say that $\sum_{i \in C} x_i \geq \lceil |C|/2 \rceil$ where C is an odd cycle of G, are valid constrains, and that they (and only they) are implied by one application of the LS lift and project method. It is therefore possible to optimize over the LP that contains the odd cycle constraints. Our goal now is to supply an independent proof to that.

1. Let G be a graph with weights (not necessarily positive) w_v on vertices, but with no cycle of negative weight. (The weight of a cycle is the sum of the weights of its vertices.) Show a polynomial time algorithm to find (one of) the lightest odd-cycle in G, that is an odd cycle that minimizes $\sum_{v \in C} w_v$ over all odd cycles in G.

   Hint: Consider the following construction of a bipartite graph H with $2n$ vertices. For every vertex $v \in V(G)$ H has two vertices v' and v''. For every edge uv in G we create two edges in H: $u'v''$ and $u''v'$; the weights of these edges are the same as the weight of the edge uv in G.

2. Provide a polynomail time algorithm to solve the LP relaxation that contains the odd cycle constrants.