Girth and Euclidean Distortion

Nathan Linial *   Avner Magen†   Assaf Naor ‡

ABSTRACT

In this paper we partially prove a conjecture that was raised by Linial, London and Rabinovich in [11]. Let G be a k-regular graph, $k \geq 3$, with girth g. We show that every embedding $f : G \to \ell_2$ has distortion $\Omega(\sqrt{g})$. The original conjecture which remains open is that the Euclidean distortion is bounded below by $\Omega(g)$. Two proofs are given, one based on semi-definite programming, and the other on Markov Type, a concept that considers random walks on metrics.

1. INTRODUCTION

Finite metric spaces and their embeddings in other metric spaces have been intensively investigated in recent years. Consequently, a growing number of algorithmic problems were solved by comprehending the metric properties of an underlying combinatorial structure, in particular graphs. For metric spaces (X, d_X) and (Y, d_Y), and an embedding $f : X \to Y$ we define the distortion of f by:

$$\mathrm{dist}(f) = \sup_{x,y\in X} \frac{d_Y(f(x), f(y))}{d_X(x,y)} \cdot \sup_{x,y\in X} \frac{d_X(x,y)}{d_Y(f(x), f(y))}.$$

*Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel. E-mail: nati@cs.huji.ac.il.
Supported in part by grants from the Israeli Academy of Sciences and the Binational Science Foundation Israel-USA.
†NEC Research Institute, Princeton, NJ. This work was done while at the Hebrew University in Jerusalem. E-mail: avner@research.nj.nec.com
‡Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel. E-mail: naor@math.huji.ac.il.
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We denote by $c_Y(X, d)$ the least distortion with which (X, d) may be embedded in Y. For $p \geq 1$ we denote $c_p(X, d) = c_{\ell_p}(X, d)$. A special case of interest is when Y is the Euclidean space ℓ_2. In this case, a fundamental result of Bourgain [5] states that $c_2(X, d) = O(\log n)$ for every n-point metric space (X, d).

One natural source for examples of metrics comes from graphs. A graph G induces a metric d_G on its vertex set, where $d_G(u, v)$ is the length of the shortest path in G joining u and v. Special families of graphs define special families of metrics, e.g. expander graph metrics are studied in [5, 14, 13], tree metrics in [2, 6, 12, 15], metrics of graphs with forbidden minors in [17, 9, 10] and many more.

Here we consider regular graphs with constant degree, and wish to study the Euclidean distortion of these graphs as a function of their girth.

In [11], Bourgain's upper bound for the Euclidean distortion was shown to be tight. In fact, the Euclidean distortion of an n point constant degree expander is $\Omega(\log n)$. A new proof of this phenomenon for the case of expanders girth $O(\log n)$ follows from the results presented in this article.

In most examples we know, metrics are far from being Euclidean, since they include "too many" triples for which the triangle inequality holds as (a near) equality. The simplest example is $K_{1,3}$ that cannot be embedded isometrically in Euclidean space, since there cannot be three geodesics between three different points that meet in their interior. For the hypercube of dimension m, there are $m!$ geodesics between every pair of antipodes, and consequently we get a large Euclidean distortion [8, 13].

It is known that for a tree T on n vertices, $c_2(T, d_T) = O(\sqrt{\log \log n})$, and this is bound is tight, as the Euclidean distortion of the complete binary tree on n vertices is $\Theta(\sqrt{\log \log n})$ (see [6, 15, 12]). Motivated by this, Linial, London and Rabinovich considered regular graphs with constant degree (bigger than 2) and with girth g. Any such graph contains isometrically a tree of depth $g/2 - 1$, which immediately gives the lower bound $c_2(G, d_G) = \Omega(\sqrt{\log g})$. Unlike the complete binary tree example, every vertex in the graph is a root of such a tree. In [11], Linial et. al. conjecture that $c_2(d_G) = \Omega(g)$. In this paper we prove $c_2(d_G) = \Omega(\sqrt{g})$.

A key ingredient (sometimes stated explicitly and often implicit in the proofs) in all the existing proofs of lower bounds for the Euclidean distortion of graphs is a Poincaré type inequality. Let G be a graph, and take any function $f : V(G) \to \ell_2$. A Poincaré type inequality bounds the average size of $\{\|f(u) - f(v)\|_2\}_{u,v \in V(G)}$ in terms of its average "gradient" $\{\|f(u) - f(v)\|\}_{uv \in E(G)}$. One over-simplified way to think of such an inequality is that it gives some bound on how common it that equality holds in the triangle inequality of triplets of points in Euclidean space. A lower bound on the distortion of an n-point k-regular expander with edge expansion $\Phi(G) = \min\left\{\frac{|E(A,V(G)-A)|}{|A|} : A \subset V(G), 1 \le |A| \le |V(G)|/2\right\}$, can be derived from the following Poincaré inequality for functions $f : V(G) \to \mathbb{R}$ (see [14]):

$$\sum_{u,v \in V(G)} |f(u) - f(v)| \le \frac{n-1}{\Phi(G)} \sum_{uv \in E(G)} |f(u) - f(v)|.$$

Let T be the complete binary tree of depth N. The following Poincaré inequality is implicit in [12], where a new proof of the estimate $c_2(T_N) = \Omega(\sqrt{\log N})$ is obtained. Denote by r the root of T, and for every integer k let $\mathcal{F}_k$ be the set of all unordered pairs of vertices $\{u, v\} \subset V(T)$ such that $d_T(u, r) = d_T(v, r)$ and $d_T(u, v) = 2^k$. Then, for every $f : V(T) \to \mathbb{R}$:

$$\sum_{k=1}^{\lfloor \log_2 N \rfloor} \sum_{\{u,v\} \in \mathcal{F}_k} 2^{-(d_T(u,r)+2^{k-1}+2k)} \cdot |f(u) - f(v)|^2 \le$$

$$C \sum_{uv \in E(T)} 2^{-d_T(u,r)} |f(u) - f(v)|^2,$$

where C is an absolute constant.

This paper further develops the above theme. We introduce two Poincaré type inequalities which are useful in the search of lower bounds for Euclidean distortion of graphs with large girth. The first is the notion of Markov type, due to K. Ball [1], which concerns the wandering of symmetric Markov chains whose state set is a metric space. We refer to section 2 for the definition. We also prove the following theorem, which can be viewed as a new Poincaré type inequality :

THEOREM 1.1. *Let H be an Euclidean space. and G be a k-regular graph, $k \ge 3$, with girth g. Fix some $1 < s < g/2$. For every $f : V(G) \to H$ the following inequality holds:*

$$\sum_{d_G(u,v)=s} \|f(u) - f(v)\|^2 \le$$

$$Cs(k-1)^{s-1} \cdot \sum_{uv \in E(G)} \|f(u) - f(v)\|^2,$$

where C is an absolute constant.

If, in addition, the graph G has a spectral gap, we can prove a stronger inequality. This leads to a new simple proof of the tightness of Bourgain's embedding theorem:

THEOREM 1.2. *Let H be an Euclidean space and G be a k regular graph, $k \ge 3$, with girth g and spectral gap $\epsilon > 0$. Fix some $1 < s < g/2$. For every $f : V(G) \to H$ the following inequality holds:*

$$\sum_{d_G(u,v)=s} \|f(u) - f(v)\|^2 \le$$

$$C(k-1)^s \cdot \frac{1 - e^{-C\epsilon s/k}}{\epsilon} \cdot \sum_{uv \in E(G)} \|f(u) - f(v)\|^2,$$

where C is an absolute constant.

We apply the above inequalities to prove our main result:

THEOREM 1.3.: *There is an absolute constant $C > 0$ such that $c_2(G) > C\sqrt{g}$ for every k-regular ($k > 2$) graph G with girth g. If, in addition, G has a spectral gap $\epsilon > 0$ then:*

$$c_2(G) \ge \frac{Cg}{\sqrt{\min\left\{g, \frac{k}{\epsilon}\right\}}}.$$

Remark: It is well known and not hard to show by probabilistic arguments that for every integer $k \ge 3$, there is an $\epsilon = \epsilon_k > 0$ and $n_0 = n_0(k)$ such that if $n \ge n_0$ and kn is even, there exist k-regular graphs of order n, spectral gap greater than ϵ and girth $\Omega(\log n)$. In view of Theorem 1.3 these graphs show that Bourgain's upper bound is tight.

This paper contains two proofs for the first part of Theorem 1.3. We first present a proof based on the notion of Markov type. Next, we prove the inequalities in Theorems 1.1 and 1.2 to deduce the full statement of Theorem 1.3. This proof is based on quadratic programming. In section 4 we discuss the interrelations between the two methods.

2. A PROOF BASED ON THE CONCEPT OF MARKOV TYPE

The first proof we present is based on the important notion of Markov type, due to K. Ball [1]. This concept is an invariant of metric spaces. It is related to other "types" that are central to the modern theory of Banach spaces. The basic assumption of this concept can also be viewed as a Poincaré inequality on metric spaces. Although we will see later that the Markov type method cannot yield the second statement in Theorem 1.3, it does give a simple and conceptual proof of the first statement. The proof we present here shows that d_G does not have Markov type 2. The lower bound on $c_2(G)$ follows from the known fact that L_2 has Markov type 2. We recall some basic theory from [1].

Let (X, d) be a metric space. A symmetric Markov chain on X is a Markov chain $\{Z_l\}_{l=0}^{\infty}$ on a state space $\{x_1, ...x_m\} \subset X$ with a symmetric transition matrix and such that Z_0 is uniformly distributed on $\{x_1, ..., x_m\}$. In other words, there is a $m \times m$ symmetric stochastic matrix $A = (a_{ij})$ such that for all $1 \le i, j \le m$ and integer $l \ge 0$, $P(Z_{l+1} = x_j | Z_l = x_i) = a_{ij}$ and $P(Z_0 = x_i) = \frac{1}{m}$. For $p > 0$ and an integer T let

$M_p(X,T)$ be the smallest constant $C>0$ such that for every symmetric Markov chain on X, $\{Z_l\}_{l=1}^{\infty}$

$$\mathbb{E}\ d^p(Z_T,Z_0)\le C^pT\mathbb{E}\ d^p(Z_1,Z_0).$$

We say that (X,d) has Markov type p if $M_p(X):=\sup_T M_p(X,T)<\infty$.In this case $M_p(X)$ is called the Markov type p constant of X.

The following proposition was shown in [1], but for the sake of completeness we prove a somewhat stronger and intuitive version of it.

PROPOSITION 2.1.: *The space L_2 has Markov Type 2.*

Proof: We first observe that the Markov type 2 property for $\mathbb{R}$ implies, by integration, the same conclusion for L_2. For symmetric Markov chains on $\mathbb{R}$ we prove the following negative correlation inequality that implies Markov type 2. Let $\{Z_l\}_{l=1}^{\infty}$ be a symmetric Markov chain with transition matrix A and state space $\{x_1,...,x_m\}\subset\mathbb{R}$. The symmetry assumption makes it intuitively plausible that Z_T-Z_{T-1} and $Z_{T-1}-Z_0$ must be negatively correlated. To prove this, notice that Z_l is uniformly distributed on $\{x_1,...,x_m\}$, for every $l\ge 0$. Since A is symmetric and stochastic, its spectrum is in $[-1,1]$, and we deduce that $(I-A)(I-A^l)$ is positive semi-definite for every l. Therefore,

$$\mathbb{E}(Z_T-Z_{T-1})(Z_{T-1}-Z_0)=$$

$$\mathbb{E}Z_TZ_{T-1}-\mathbb{E}Z_TZ_0-\mathbb{E}Z_{T-1}^2+\mathbb{E}Z_{T-1}Z_0=$$

$$=\frac{1}{m}\sum_{i,j=1}^{m}(A)_{ij}x_ix_j-\frac{1}{m}\sum_{i,j=1}^{m}(A^T)_{ij}x_ix_j-$$

$$\frac{1}{m}\sum_{i=1}^{m}x_i^2+\frac{1}{m}\sum_{i,j=1}^{m}(A^{T-1})_{ij}x_ix_j=$$

$$-\frac{1}{m}\langle(I+A^T-A-A^{T-1})x,x\rangle=$$

$$-\frac{1}{m}\langle(I-A)(I-A^{T-1})x,x\rangle\le 0,$$

where $x=(x_1,...,x_m)$. Hence :

$$\mathbb{E}(Z_T-Z_0)^2=\mathbb{E}(Z_{T-1}-Z_0+Z_T-Z_{T-1})^2=$$

$$\mathbb{E}(Z_{T-1}-Z_0)^2+2\mathbb{E}(Z_T-Z_{T-1})(Z_{T-1}-Z_0)+\mathbb{E}(Z_T-Z_{T-1})^2$$

$$\le\mathbb{E}(Z_{T-1}-Z_0)^2+\mathbb{E}(Z_1-Z_0)^2.$$

By summing this inequality over $T=1,...,N$, we deduce that the real line has Markov type 2 with constant 1. ∎

The following simple consequence of the above analysis will be useful for us:

COROLLARY 2.2.: *For every metric space (X,d), $c_2(X,d)\ge M_2(X)$.*

Proof: Fix some embedding $f:X\to L_2$ such that $1/D\le\|f(x)-f(y)\|/d(x,y)\le 1$ for every $x,y\in X$. For every symmetric Markov chain $\{Z_l\}_{l=0}^{\infty}$ on X, the Markov type 2 property of L_2 applied to the Markov chain $\{f(Z_l)\}_{l=0}^{\infty}$ gives:

$$\frac{1}{D^2}\mathbb{E}\ d^2(Z_T,Z_0)\le\mathbb{E}\|f(Z_T)-f(Z_0)\|^2\le$$

$$T\mathbb{E}\|f(Z_1)-f(Z_0)\|^2\le T\mathbb{E}\ d^2(Z_1,Z_0),$$

so that $D\ge M_2(X)$. ∎

The first assertion in Theorem 1.3 follows from the following:

PROPOSITION 2.3. *Let G be a k-regular graph with girth g. Then*

$$M_2(G)\ge\frac{k-2}{k}\sqrt{\left\lceil\frac{g}{2}-1\right\rceil}.$$

Proof: Consider the symmetric Markov chain $\{Z_t\}_{t=0}^{\infty}$ that corresponds to the canonical random walk on G. Namely, Z_0 is uniformly distributed on $V(G)$ and $P(Z_{t+1}=v|Z_t=u)$ equals $\frac{1}{k}$ if u and v are neighbors and 0 otherwise. Note that every vertex $v\in V(G)$ is the root of a k-regular tree of depth $g/2$ (or more precisely, as a metric space, each ball of radius smaller than $g/2$ in G is isometric to such a tree, whose root is the center of the ball). As long as $T<g/2$, each step of the random walk $\{Z_t\}_{t=0}^{T}$ moves away from Z_0 with probability at least $\frac{k-1}{k}$ (if $Z_t=Z_0$ then this probability is 1) and towards it with probability at most $\frac{1}{k}$. In other words, as long as $T<g/2$, the random walk has a positive drift away from Z_0. To quantify this, for every $1<T<g/2$ we have:

$$\mathbb{E}\ d_G(Z_T,Z_0)\ge$$

$$\frac{k-1}{k}\big(\mathbb{E}\ d_G(Z_{T-1},Z_0)+1\big)+\frac{1}{k}\big(\mathbb{E}\ d_G(Z_{T-1},Z_0)-1\big)=$$

$$\mathbb{E}\ d_G(Z_{T-1},Z_0)+\frac{k-2}{k}.$$

Hence, for every $T<g/2$:

$$\mathbb{E}\ d_G^2(Z_T,Z_0)\ge\big(\mathbb{E}\ d_G(Z_T,Z_0)\big)^2\ge\left(\frac{k-2}{k}\right)^2T^2.$$

On the other hand:

$$\mathbb{E}\ d_G^2(Z_T,Z_0)\le M_2(G)^2T\mathbb{E}\ d_G^2(Z_1,Z_0)=M_2(G)^2T.$$

The proposition follows by taking $T=\lceil\frac{g}{2}-1\rceil$. ∎

Now, together with Corollary 2.2 this proposition implies that $c_2(G)\ge C\sqrt{g}$, where C is an absolute constant not depending on k.

3. BOUNDING THE DISTORTION VIA SEMI DEFINITE PROGRAMMING

In what follows G is a k regular graph with girth g.

Semi-Definite Programming has proved to be a central tool in establishing lower bounds on $c_2(\cdot)$. Our proof of Theorems 1.1 and 1.2, and our second proof of Theorem 1.3 are all based on this point of view, together with an analysis of the algebraic properties of the graphs in question. We first present the necessary background.

Let PSD_n be the cone of positive semi-definite symmetric $n \times n$ matrices. Define $\mathcal{B}_n$ to be $\{Q \in PSD_n | Q\vec{1} = 0\}$, and let

$$\delta(Q,d) = \left(\frac{\sum_{i,j:Q_{i,j}>0} d^2(i,j) Q_{i,j}}{\sum_{i,j:Q_{i,j}<0} d^2(i,j) |Q_{i,j}|} \right)^{\frac{1}{2}},$$

if the denominator is not 0, and 1 otherwise.

The following lemma gives a formula for c_2.

Proposition 3.1.: *(Linial, London, Rabinovich [11]) Suppose X is finite, $|X| = n$, then*

$$c_2(X,d) = \sup_{Q \in \mathcal{B}_n} \delta(Q,d)$$

In order to make use of the algebraic properties of G, we turn to some background on the following very useful concept.

3.1 Geronimus Polynomials

Let G be a k-regular graph with girth g and let A be its adjacency matrix. We define $A^{(t)}$ as G's distance t matrix. Namely, $A^{(t)}_{i,j} = 1$ if the distance $d_G(i,j) = t$ and 0 otherwise.

There exist polynomials P_t, such that P_t has degree t and $P_t(A) = A^{(t)}$ for every $t < g/2$. The conditions that define these polynomials easily translate to a simple recurrence relation. Clearly $P_0(x) = 1$, and $P_1(x) = x$. Note that $A^{(t)} - A \cdot A^{(t-1)}$ equals $-kA^{(t-2)} = -k \cdot I$ for $t = 2$, and $-(k-1)A^{(t-2)}$ for $2 < t < g/2$. Therefore,

$$P_2(x) = xP_1(x) - kP_0(x) = x^2 - k,$$

and

$$P_t(x) = xP_{t-1}(x) - (k-1)P_{t-2}(x) \quad \text{for every } t > 2.$$

These polynomials are often called in the literature "Geronimus Polynomials", a name that we adopt. In Figure 1 the first even Geronimus polynomials are shown, in the relevant range of $[-k,k]$. Basic facts about Geronimus polynomials can be found in [3, 18] (but note the different normalization used in these references). To make this discussion self contained, we briefly review some of the necessary facts and sketch their proofs.

In order to understand the analytical properties of the Geronimus polynomials, one first solves the recursion and finds an explicit formula for them. The following trigonometric expression is obtained:

$$\forall t > 0 \quad P_t(2\sqrt{k-1}\cos\theta) =$$

$$(k-1)^{t/2-1} \frac{(k-1)\sin((t+1)\theta) - \sin((t-1)\theta)}{\sin\theta} \quad (1)$$

To verify this identity, check the cases $t = 1, 2$ and note that for $t > 2$ the recursion relation holds.

Our next observation is that all the roots of P_t are real and they all lie between $-2\sqrt{k-1}$ and $2\sqrt{k-1}$. This can be derived from the general theory of orthogonal polynomials (e.g. [19]), but we provide a direct proof. By identity 1 it suffices to show there are t distinct real values of θ in $[0,\pi)$ for which the above expression vanishes. Indeed, define $\theta_q = (\frac{\pi}{2} + q\pi)/(t+1)$

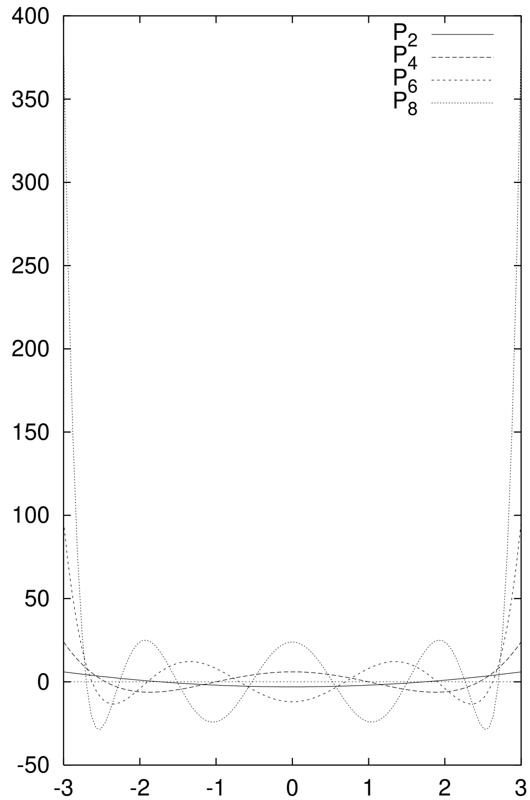

Figure 1: The first (even) Geronimus polynomials with $k = 3$.

for $q = 0, 1 \ldots, t-1$. Now, it is not hard to see that $P_t(2\sqrt{k-1}\cos\theta_q)$ is positive for even q, and negative for q odd. Therefore, there is a zero for some value of θ between θ_q and θ_{q+1}, yielding t zeros in the desired interval.

The last two facts that we need are easily verified by induction:

$$P_t(k) = k(k-1)^{t-1} \quad \forall t > 0,$$

and

$$P'_t(k) = (k-2)^{-2}\big(t(k-1)^{t+1} - 2(k-1)^t - t(k-1)^{t-1} + 2\big).$$

3.2 Technical lemmas

Even though the Geronimus polynomials are not convex throughout $[-k,k]$, we now prove an inequality that reflects the fact that their non-convexity is restricted to the range $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. The proof uses the classical Markov inequality (see [4]): $\|P'\|_{L_\infty[-1,1]} \le n^2 \|P\|_{L_\infty[-1,1]}$ for every real polynomial P of degree n, where $\|f\|_{L_\infty[-a,a]} = \sup_{|x| \le a} |f(x)|$. A more direct proof can be given by differentiating formula 1, but use of Markov's inequality eliminates a tedious calculation which leads, essentially, to the same estimate.

Lemma 3.2.: *Let $s \ge 40$ be an even integer. For* *every* $\epsilon > 0$ *and* $x \in [-k, k-\epsilon]$,

$$\frac{P_s(k) - P_s(k-\epsilon)}{\epsilon} \geq \frac{P_s(k) - P_s(x)}{k-x}.$$

Proof: Define:

$$f(x) = \frac{P_s(k) - P_s(x)}{k-x}.$$

We need to show that f is non-decreasing on $[-k, k]$. By taking a derivative of the right hand side and expanding, this follows from the claim that for all $x \in [-k, k]$,

$$h(x) := P_s(x) + (k-x)P_s'(x) \leq P_s(k).$$

Note that $h(k) = P_s(k)$ and, since P_s is an even function (for s even), it follows that $h(-k) = P_s(k) - 2kP_s'(k) < P_s(k)$ (since $P_s'(k) > 0$). It is therefore enough to show that $h(x_0) \leq P_s(k)$ whenever $h'(x_0) = 0$. Now, $h'(x) = (k-x)P_s''(x)$, so that the zeros of h' coincide with the zeros of P_s''. Since P_s has all its roots in the interval, $[-2\sqrt{k-1}, 2\sqrt{k-1}]$, the same holds for P_s''. It therefore suffices to show that $h(x) \leq P_s(k)$ throughout the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. Every point in this interval has the form $x = 2\sqrt{k-1}\cos\theta$ for some $0 \leq \theta \leq \pi$. Using the trigonometric expression 1 we get:

$$P_s(x) = P_s(2\cos\theta\sqrt{k-1}) =$$

$$(k-1)^{s/2-1}\frac{(k-1)\sin((s+1)\theta) - \sin((s-1)\theta)}{\sin\theta}.$$

It is easily verified that $\sin(r\alpha) \leq r\sin\alpha$ for $\alpha \in [0, \pi)$ and $r \geq 1$. Therefore,

$$\|P_s\|_{L_\infty[-2\sqrt{k-1}, 2\sqrt{k-1}]} \leq$$

$$(k-1)^{s/2-1}\big((k-1)(s+1) + (s-1)\big).$$

Markov's inequality implies that

$$\|P_s'\|_{L_\infty[-2\sqrt{k-1}, 2\sqrt{k-1}]} \leq$$

$$\frac{s^2}{2\sqrt{k-1}} \cdot \|P_s\|_{L_\infty[-2\sqrt{k-1}, 2\sqrt{k-1}]},$$

so that:

$$\|h\|_{L_\infty[-2\sqrt{k-1}, 2\sqrt{k-1}]} \leq$$

$$\|P_s\|_{L_\infty[-2\sqrt{k-1}, 2\sqrt{k-1}]} + 2k\|P_s'\|_{L_\infty[-2\sqrt{k-1}, 2\sqrt{k-1}]} \leq$$

$$(k-1)^{s/2-1}\big((k-1)(s+1) + (s-1)\big)\left(1 + \frac{ks^2}{\sqrt{k-1}}\right) \leq$$

$$k(k-1)^{s-1} = P_s(k).$$

It is not hard to verify the last inequality for every $k \geq 3$ and $s \geq 40$. ∎

We need one more fact concerning Geronimus polynomials:

LEMMA 3.3.: *For every integer* s *and* $\epsilon \leq k/20$,

$$P_s(k-\epsilon) \geq k(k-1)^{s-1} \cdot e^{-150\epsilon s/k}.$$

Proof: Let $y_1, \ldots, y_s$ be the roots of P_s. By the mean value theorem there is some $a \in (k-\epsilon, k)$ such that:

$$\log\left[\frac{P_s(k)}{P_s(k-\epsilon)}\right] = \epsilon\frac{P_s'(a)}{P_s(a)} =$$

$$= \epsilon \cdot \sum_{i=1}^{s}\frac{1}{a-y_i} \leq \frac{\epsilon s}{k-\epsilon-2\sqrt{k-1}} \leq \frac{150\epsilon s}{k},$$

where we have used the facts that $y_1, \ldots, y_n \in [-2\sqrt{k-1}, 2\sqrt{k-1}]$, $\epsilon \leq k/20$ and $k \geq 3$. This yields the required result since $P_s(k) = k(k-1)^{s-1}$. ∎

COROLLARY 3.4. *Let* $s \geq 40$ *be an even integer and* $0 < \epsilon \leq k$. *Then:*

$$1 - \frac{P_s(k-\epsilon)}{P_s(k)} \leq C\big(1 - e^{-C\epsilon s/k}\big),$$

for some absolute constant $C > 0$.

Proof: When $\epsilon \leq k/20$ this follows from Lemma 3.3. When $\epsilon > k/20$ the right hand side is bounded from below by an absolute constant. The left hand side is at most 2, since the minimum of P_s is attained in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$, and the bound obtained in the proof of Lemma 3.2 implies in particular that $P_s(x) \geq -P_s(k)$ for every x in this interval. ∎

3.3 Proof of Theorems 1.1 and 1.2

Let G be a k-regular graph with girth g and fix an integer $1 < s < g/2$. For every two vertices $u, v \in V(G)$ such that $d_G(u,v) = s$ there is a unique path of length s joining u and v. In other words, there is a unique set of vertices $\{w_{u,v}(i)\}_{i=0}^s \subset V(G)$ such that $w_{u,v}(0) = u$, $w_{u,v}(s) = v$ and for every $i \geq 1$, $[w_{u,v}(i-1), w_{u,v}(i)] \in E(G)$. Each edge $e \in E(G)$ appears in exactly $s(k-1)^{s-1}$ such paths. Hence, for every metric space (X, d_X) and every mapping $f : V(G) \to X$

$$\sum_{d_G(u,v)=s} d_X^2(f(u), f(v)) \leq$$

$$\sum_{d_G(u,v)=s}\left(\sum_{i=1}^{s} d_X\big(f(w_{u,v}(i-1)), f(w_{u,v}(i))\big)\right)^2 \leq$$

$$\leq \sum_{d_G(u,v)=s} s\sum_{i=1}^{s} d_X^2\big(f(w_{u,v}(i-1)), f(w_{u,v}(i))\big) =$$

$$s^2(k-1)^{s-1}\sum_{uv \in E(G)} d_X^2(f(u), f(v)).$$

It follows that the inequalities of Theorems 1.1 and 1.2 are trivially true for bounded s, so that we may clearly assume that $s \geq 40$. By a similar argument, we can also assume that s is even. In addition, in both statements it is clearly enough to assume that f maps G into $H = \mathbb{R}$.

We begin with the the proof of Theorem 1.2. Let G satisfy the conditions of Theorem 1.2 and consider the following matrix:

$$Q = \alpha I - A + \beta A^{(s)},$$

where $40 \le s < g/2$ is an even integer, and:

$$\alpha = k - \frac{\epsilon P_s(k)}{P_s(k) - P_s(k-\epsilon)},$$

$$\beta = \frac{\epsilon}{P_s(k) - P_s(k-\epsilon)}.$$

We claim that $Q \in \mathcal{B}_n$. Now, Q is clearly symmetric. Also, since $A\vec{1} = k\vec{1}$, $A^{(s)}\vec{1} = P_s(k)\vec{1}$ so that $Q\vec{1} = 0$. Other than the eigenvalue k, the spectrum of A is in $[-k, k-\epsilon]$, so Q will be shown to be positive semi-definite if for all $x \in [-k, k-\epsilon]$

$$\alpha - x + \beta P_s(x) \ge 0,$$

which is precisely the statement of Lemma 3.2.

Now, for every $f : V(G) \to \mathbb{R}$ we get that:

$$0 \le \langle Qf, f\rangle =$$

$$= \alpha \sum_{u \in V(G)} f(u)^2 - \sum_{u,v \in V(G)} A_{uv} f(u) f(v) +$$

$$\beta \sum_{u,v \in V(G)} A^{(s)}_{uv} f(u) f(v) =$$

$$= \frac{1}{2} \sum_{uv \in E(G)} |f(u) - f(v)|^2 - \frac{\beta}{2} \sum_{d_G(u,v)=s} |f(u) - f(v)|^2,$$

where in the last equality we have used the fact that $\alpha - k + \beta P_s(k) = 0$. Hence:

$$\sum_{d_G(u,v)=s} |f(u) - f(v)|^2 \le \frac{1}{\beta} \sum_{uv \in E(G)} |f(u) - f(v)|^2 =$$

$$= \frac{k(k-1)^{s-1}}{\epsilon} \left[1 - \frac{P_s(k-\epsilon)}{P_s(k)}\right] \sum_{uv \in E(G)} |f(u) - f(v)|^2$$

$$\le C(k-1)^s \cdot \frac{1 - e^{-C\epsilon s/k}}{\epsilon} \sum_{uv \in E(G)} |f(u) - f(v)|^2,$$

where we have used Corollary 3.4.

The proof of Theorem 1.1 runs along the same lines. We return to the above construction and let $\epsilon \to 0$. This yields a matrix:

$$\widetilde{Q} = \left[k - \frac{P_s(k)}{P_s'(k)}\right] I - A + \frac{1}{P_s'(k)} A^{(s)},$$

which by continuity is in $\mathcal{B}_n$ as well. Arguing as above, we get the following inequality:

$$\sum_{d_G(u,v)=s} |f(u) - f(v)|^2 \le P_s'(k) \sum_{uv \in E(G)} |f(u) - f(v)|^2$$

$$= \frac{s(k-1)^{s+1} - 2(k-1)^s - s(k-1)^{s-1} + 2}{(k-2)^2} \cdot$$

$$\sum_{uv \in E(G)} |f(u) - f(v)|^2,$$

which implies the required result. $\square$

3.4 Proof of Theorem 1.3

We now deduce Theorem 1.3 from Theorems 1.1 and 1.2. Let G be a k-regular graph ($k \ge 3$) with girth g. Take any embedding $f : V(G) \to \ell_2$ such that for every $u, v \in V(G)$

$$\frac{1}{D} \le \frac{\|f(u) - f(v)\|}{d_G(u,v)} \le 1.$$

Set $s = [g/2] - 1$. Since there are $k(k-1)^{s-1}$ vertices of distance s from a fixed vertex we get that:

$$\sum_{d_G(u,v)=s} \|f(u) - f(v)\|^2 \ge \frac{s^2 k(k-1)^{s-1} |V(G)|}{D^2},$$

and

$$\sum_{uv \in E(G)} \|f(u) - f(v)\|^2 \le k|V(G)|.$$

Theorem 1.1 now gives that:

$$\frac{s^2 k(k-1)^{s-1} |V(G)|}{D^2} \le Cs(k-1)^s |V(G)|,$$

so that $c_2(G) \ge c'\sqrt{g}$. If in addition we assume that G has a spectral gap ϵ, this reasoning gives:

$$c_2(G) \ge c' g \sqrt{\frac{\epsilon/k}{1 - e^{-Cg\epsilon/k}}} \ge \frac{c'' g}{\sqrt{\min\{g, \frac{k}{\epsilon}\}}},$$

which finishes the proof of Theorem 1.3.

4. THE RELATIONSHIP BETWEEN THE TWO METHODS

The proof of the Markov type 2 property of Euclidean space in [1], sheds some light on the connection between the two methods. In Proposition 3.1 we seek the matrix Q with maximal $\delta(Q, d)$ over all $Q \in \mathcal{B}_n$. We will show how to view Corollary 2.2 as a restriction of Proposition 3.1, in the sense that only a subset of the matrices in $\mathcal{B}_n$ can be used for the lower bound.

Let B be the symmetric stochastic transition matrix defining the Markov chain, and let $\gamma \in (0, 1)$. Now define

$$R = \frac{2\gamma - 1}{\gamma} I - B + \frac{(1-\gamma)^2}{\gamma} \sum_{l \ge 0} (\gamma B)^l$$

The spectrum of B clearly determines that of R: If λ is an eigenvalue of B, then $\frac{2\gamma-1}{\gamma} - \lambda + \frac{(1-\gamma)^2}{\gamma(1-\gamma\lambda)}$ is an eigenvalue of R. This expression is nonnegative for $\lambda \in [-1, 1]$, and 0 for $\lambda = 1$. This clearly means that $R \in \mathcal{B}_n$. If $x_1, \ldots, x_n$ are vectors in an Euclidean space then,

$$\sum_{i,j} R_{i,j} \|x_i - x_j\|^2 \le 0,$$

and so

$$(1-\gamma) \sum_{ij} (\sum_l (\gamma B)^l)_{ij} \|x_i - x_j\|^2 \le$$

$$\gamma \sum_{ij} B_{ij} \|x_i - x_j\|^2.$$

As Ball shows by taking $\gamma = 1 - \frac{1}{m}$, this is equivalent to the fact that Euclidean space has Markov type 2. To be more precise, this is the original definition he gives to the metric property "having a Markov type 2".

Thus the approach via Markov type can be viewed as a specialized version of the semidefinite programming method. This method is incomplete in that not every matrix in $\mathcal{B}_n$ is attained from the above transformation of symmetric stochastic matrices. This method is still very useful, in that it allows one to draw on geometric and probabilistic intuitions. In contrast, the semidefinite approach greatly depends on successful clever guesses of the matrix Q.

The Poincaré inequality method can also be viewed as a restriction of the family $\mathcal{B}_n$. Indeed, if G is a graph and B is any matrix such that $B_{uv} = 0$ whenever u and v are not neighbors in G, then for any matrix C and $\alpha \in \mathbb{R}$, the fact that the matrix $\alpha I - B + C$ is in $\mathcal{B}_n$ is equivalent to the Poincaré inequality:

$$\sum_{u,v\in V(G)} C_{uv}|f(u)-f(v)|^2 \leq \sum_{uv\in E(G)} B_{uv}|f(u)-f(v)|^2,$$

for every $f : V(G) \to \mathbb{R}$. Thus, all the existing lower bounds for $c_2(\cdot)$ are based on the study of the above subset of $\mathcal{B}_n$, which leads to geometrically intuitive, Poincaré inequality reasons for non-embeddability of graphs in Euclidean space. Hence, in a sense, the full strength of Proposition 3.1 is yet to be fully exploited.

The second part of Theorem 1.3 cannot be derived from Markov type considerations. Indeed, let G be any graph. If $\{Z_k\}_{k=0}^{\infty}$ is a symmetric Markov chain on G such that $P(Z_{k+1} = u | Z_k = u) = 0$ for every $u \in V(G)$ then clearly $\mathbb{E}\, d_G^2(Z_T, Z_0) \leq \mathrm{diam}(G)^2 \mathbb{E}\, d_G^2(Z_1, Z_0)$ and by the triangle inequality $\mathbb{E}\, d_G^2(Z_T, Z_0) \leq T^2 \mathbb{E}\, d_G^2(Z_1, Z_0)$. Hence, for every T,

$$M_2(G) \leq \min\left\{\frac{\mathrm{diam}(G)}{\sqrt{T}}, T\right\},$$

which implies that $M_2(G) = O(\mathrm{diam}(G)^{2/3})$, while our second proof of Theorem 1.3 showed that there are graphs with $c_2(G) = \Omega(\mathrm{diam}(G))$.

5. DISCUSSION

An interesting problem that still remains open is the determination of the worst possible behavior of $c_2(G)$ over all k-regular, $k \geq 3$, graphs with girth g. Our present methods seem to break at $\sqrt{g}$. On the other hand if k-regular graphs, G, with girth g and $c_2(G) = O(\sqrt{g})$ exist, they cannot be expanders in view of the known lower bound for the Euclidean distortion of expanders. This observation rules out most known constructions for graphs with large girth. Although examples of k-regular graphs with large girth which are not expanders are known, these specific examples seem highly complicated and far from Euclidean.

It is interesting to note that we know literally nothing about the effect of high girth on ℓ_1 embeddings. Let G be a graph in which all vertex degrees are > 2 and suppose that G has girth g. Can we derive any lower bound on $c_1(G)$ that tends to ∞ with g ? The analogous question for c_2 was relatively easy, since G contains an isometric copy of a complete binary tree of depth $g/2$, and such trees do not embed into ℓ_2 with constant distortion. On the other hand, they do embed into ℓ_1 isometrically. Hence the problem.

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