PSPACE-Completeness

CSC 463

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PSPACE-Completeness: Basics

- A language/decision problem A is **PSPACE**-Complete if:
  - $A \in PSPACE$
  - There is a polynomial time reduction $B \leq_p A$ for any $B \in PSPACE$.

Theorem

Let **TQBF** *be the problem of deciding if a fully-quantified Boolean formula ϕ is true or false.* **TQBF** *is* **PSPACE**-*Complete.*

- **Examples:** $\forall x \exists y (x \vee y)$ is true, but $\forall x \exists y (x \wedge y)$ is false.
- Proof techniques for showing **TQBF** is **PSPACE**-Complete similar to that of the Cook-Levin theorem for NP-Completeness of SAT.

PSPACE-Completeness and Games

- A game involves two players performing actions according to some specified rules until one of the players achieves some goal to win the game.
- Studied in artificial intelligence/machine learning and economics.
- Examples: Tic-Tac-Toe, Go, Chess, Checkers, etc.
- A player in a game has a **winning strategy** if no matter what the other player does, the player has a way to win.

PSPACE-Completeness and Games

- Determining if a player has a winning strategy in many games is PSPACE-Complete.
- Intuition: determining if someone has a winning strategy is like seeing if a TQBF formula is true. If a_i are Player 1s actions and b_i are Player 2s' actions then Player 1 has a winning strategy if

  $$\exists a_1 \forall b_1 \exists a_2 \forall b_2 \ldots W(a_1, b_1, a_2, b_2, \ldots)$$

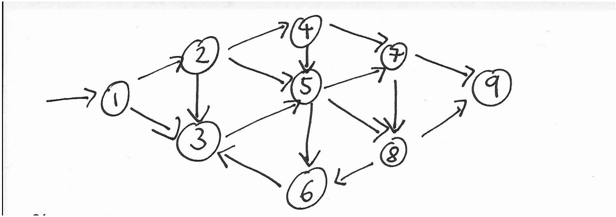
  is true where W is the winning condition of the game depending on the players' actions. (Here a_i, b_i not Boolean but taken from some other domain.)

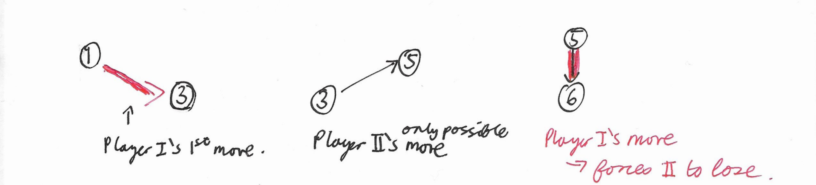
Geography Game

- We will study the generalized geography game.
- There are two players. Player 1 starts by saying the name of a city c. Player 2 then follows by saying the name of a city that begins with the last letter of c. This continues until some player cannot think of another city or repeats one already said, in which case the other player wins.
- Example gameplay:
  Toronto $\rightarrow$ Oakville $\rightarrow$ Edmonton $\rightarrow$ New Westminster $\rightarrow$ Rimouski $\rightarrow$ Iqualuit $\rightarrow \dots$

Geography Game (GG)

- We model the game as a directed graph G. The players take turns choosing vertices from G such that the vertices form a simple path (no vertices repeated). A player loses when they are unable to continue the path. We want to decide if Player 1 has a winning strategy for geography on G starting at some initial vertex b.





Geography Game

- $GG \in PSPACE$: Given a graph G and a starting vertex b, the algorithm $test_{gg}$(G,b) checks if Player 1 has a winning strategy:
  1. If the outdegree of b is 0, Player 1 immediately loses so return False.
  2. Otherwise, let $b_1, \ldots, b_k$ be the vertices b points to in G and G' be G with b and its incident edges removed.
  3. Check $test_{gg}(G', b_i)$ for $i = 1, \ldots, k$. If all return True, Player 2 has a winning strategy, otherwise Player 1 has a winning strategy.
- This takes $O(n)$ space where n is the number of vertices in the graph.

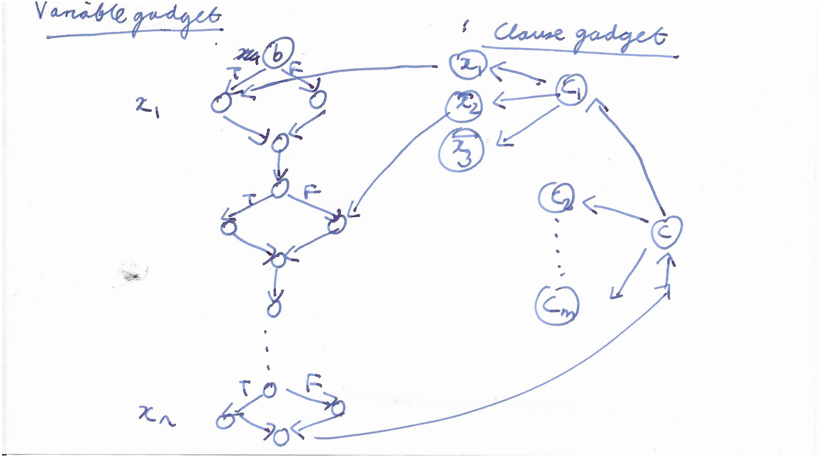
Geography Game

- Now for hardness we need to argue that GG is PSPACE-hard. We do this by providing a reduction $TQBF \leq_p GG$.
- We assume that we are given a formula ϕ for alternating quantifiers:

$$\exists x_1 \, \forall x_2 \, \exists x_3 \, \ldots \, \exists x_k \psi(x_1, \ldots, x_k)$$

  where k is odd, ψ is a 3-CNF propositional formula and $Q \in \{\exists, \forall\}$ and have to construct a graph G where Player I has a winning strategy iff the formula ϕ is true.
- This proof is somewhat similar to the proof that Hamiltonian path is NP-Complete.

Geography Game: Picture of the Reduction



Observation: You can pick the truth values of variables $x_1, x_3, x_5, \ldots,$ and the opposing player picks truth values of $x_2, x_4, x_6, \ldots$.

Correctness of the reduction

- After a truth assignment has been picked, you must visit vertex c, and your opponent then picks some clause c_i.
- If the truth assignment satisfies the formula, ψ, you can pick the variable that makes c_i true to make your opponent lose.
- Otherwise, your opponent can pick c_i that falsfies ψ, and then you are then forced to revisit an already visited vertex.
- Truth assignments for $x_1, x_3, \ldots$ that make the formula true **regardless** of what is picked for $x_2, x_4, \ldots$ exist iff the TQBF formula ϕ was true.

Additional PSPACE-Complete problems

- Regular expressions equivalence: Given two regular expressions R, S is it the case that $L(R) = L(S)$?
- Robot motion planning: Given a mathematical description of a robot (as a polygon in $\mathbb{R}^2, \mathbb{R}^3$) and obstacles their environment (also described by polygons), is there a path from some initial position to the final position in the environment that avoids the obstacles in their environment?
- Interactive proofs: An interactive proof informally is like a game except winning can be determined randomly rather than deterministically. An interactive proof can be constructed for problem in PSPACE (Shamir 1992).