Tight Hardness Results for Minimizing Discrepancy

Moses Charikar Alantha Newman *Aleksandar Nikolov*

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Discrepancy of $\{S_1, \ldots, S_M\}$: minimum of (1) over all assignments.

Example

What is the discrepancy of the *five-cycle*?

$$
\begin{aligned}
S_1 &= \{x_1, x_2\} \\
S_2 &= \{x_2, x_3\} \\
S_3 &= \{x_3, x_4\} \\
S_4 &= \{x_4, x_5\} \\
S_5 &= \{x_1, x_5\}
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2: No matter how we alternate -1 and $+1$, one edge will be monochromatic.

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Bansal[Ban10]: algorithm to find the assignment *in polynomial time*.

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Consequence: *Discrepancy cannot be approximated to within any multiplicative factor.*

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Theorem (**Main Theorem)**

Let $\{S_1, \ldots, S_M\}$ be a set system on N elements and $M = O(N)$ sets. It is NP-hard to distinguish between the following cases:

1. *the set system has discrepancy 0*
2. *the set system has discrepancy $\Omega(\sqrt{N})$.*

Notation

Set system $\Leftrightarrow$ *incidence matrix* A (A_{j*}: indicator vector of S_j).

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We will need a related notion of discrepancy, ℓ_2^2 discrepancy:

$$D_2^2(A) = \min_{x \in \{\pm 1\}^N} \|Ax\|_2^2.$$

Fact

$$D_\infty^2(A) \geq \frac{D_2^2(A)}{M} \qquad \Rightarrow \qquad D_\infty(A) \;\geq\; \sqrt{\frac{D_2^2(A)}{M}}.$$

We prove:

Theorem

Given an $M \times N$ 0-1 matrix A with $M = O(N)$, it is NP-hard to distinguish between the cases

1. $D_2^2(A) = 0$ *(*$\Rightarrow D_\infty = 0$*),*
2. $D_2^2(A) \geq \Omega(N^2)$ *(*$\Rightarrow D_\infty = \Omega(\sqrt{N})$*).*

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This theorem implies the main theorem.

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- Simple composition: hardness for multisets;
- Decomposing MAX-2-2-SET-SPLITTING + simple composition: hardness for sets.

Multisets

We start with an easier theorem:

Theorem

Given an $M \times N$ matrix B with $M = O(N)$ and entries in $\{0, \ldots, b\}$, where b is a constant, it is NP-hard to distinguish between the cases:

1. $\exists y \in \{-1, 1\}^N$ *for which* $||By||_2^2 = 0$;
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Corresponds to discrepancy of multisets.

Set Splitting

We reduce from:

MAX-2-2-SET-SPLITTING: given a set system of m sets on n elements, each consisting of 4 elements, and each element appearing in $\leq b$ sets, b a constant ($m = O(n)$).

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C: incidence matrix of a MAX-2-2-SET-SPLITTING instance. [Gur03]:it is NP-hard to distinguish between:

1. There is an assignment such that each set has discrepancy 0 ($D_2^2(C) = 0$).
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We need to amplify the 0 *vs* $\Omega(n)$ *gap to* 0 *vs* $\Omega(n^2)$.

Hadamard Matrices

Hadamard matrices are ± 1 symmetric matrices whose columns and rows are pairwise orthogonal. The easiest to construct are:

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; H_n = \begin{pmatrix} H_{n/2} & H_{n/2} \\ H_{n/2} & -H_{n/2} \end{pmatrix}.$$

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Lemma

Let W be a $k \times k$ matrix as defined above. Let $x \in \mathbb{R}^k$ be a vector such that $\sum_{i>1} x_i^2 = \Omega(k)$. Then $\|Wx\|_2^2 = \Omega(k^2)$.

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A slight strengthening of the lower bound for ± 1 assignments [Cha91].

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- If $D_2^2(C) = \Omega(n)$, $\forall y : \|By\|_2^2 = \|W(Cy)\|_2^2 = \Omega(n^2)$.

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Reduction is to *multisets* because an element in a MAX-2-2-SET-SPLITTING instance can appear in *more than one set*.

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Workaround:

- partition the sets so that *in each partition each element appears once*;
- apply the reduction to each partition.

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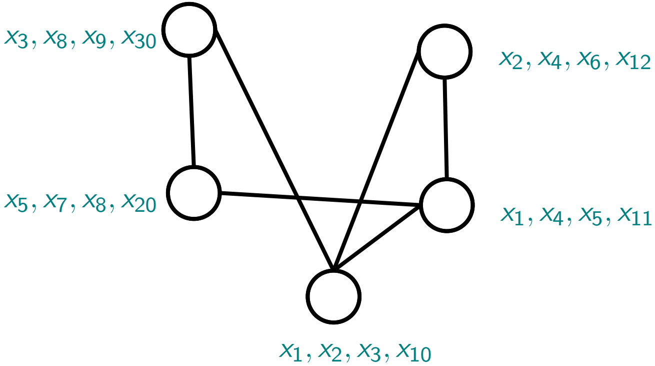
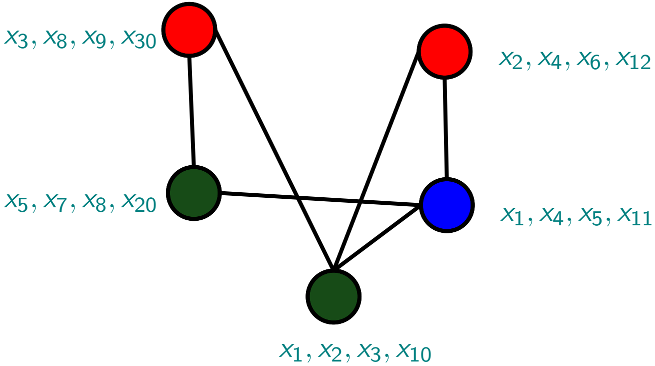
- two vertices are connected if they share an element.
- G is constant degree, i.e. has constant chromatic number. There is a *constant number* of color classes, each containing *non-overlapping sets*.


x_3, x_8, x_9, x_{30}
x_2, x_4, x_6, x_{12}
x_5, x_7, x_8, x_{20}
x_1, x_4, x_5, x_{11}
x_1, x_2, x_3, x_{10}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- Since in each partition each element appears once, A is 0-1.
- When $D_2^2(C) = 0$, $D_2^2(A) = 0$.
- When $D_2^2(C) = \Omega(n)$, then for any assignment y, there exists a partition with incidence matrix C' such that $\|C'y\|_2^2 \geq \Omega(1)\|Cy\|_2^2 = \Omega(n)$ (by averaging). Then $\|Ay\|_2^2 \geq \|WC'y\|_2^2 = \Omega(n^2)$.

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Using this idea we prove a *tight hardness result* for set systems with *bounded shatter function*.

Open Problems

Our hardness results are for set systems with $M = O(N)$. Can we show hardness results for other regimes of M?

Other notions of discrepancy exist (e.g. hereditary discrepancy, linear discrepancy). What is the computational complexity of those notions of discrepancy?

Thank you!

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