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| **CSC2414: Discrepancy Theory in Computer Science** | Fall 2015 |

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# 1 Higher Dimensional Discrepancy

Let us begin by recalling some notions from the previous lecture. We analyzed the two-dimensional *star discrepancy*, denoted $D(n, \mathcal{C}_2)$, where $\mathcal{C}_2 = \{C_y \mid y \in [0,1)^2\}$ is the collection of all two-dimensional corners

$$C_y = \{x \in [0,1)^2 \mid 0 \leq x_1 \leq y_1, 0 \leq x_2 \leq y_2\}.$$

In particular, we saw that (up to constants) we have

$$D(n, \mathcal{C}_2) \cong D(n, \mathcal{R}_2) \cong \log n \tag{1}$$

where $\mathcal{R}_2$ is the collection of all rectangles in $[0,1)^2$. There are several ways to prove the above relations: we focused on an upper bound method employing *van der Corput sequences*, and an easier lower bound of $\Omega(\sqrt{\log n})$ originally due to Roth [4]. An immediate application of (1) is effective integral estimation for well-behaved[1] $f : [0,1] \to \mathbf{R}$ via the *Koksma-Hlawka inequality*: namely, for any positive integer $N$ there exists a sequence $u_1, u_2, \ldots, u_N \in [0,1)$ such that for any well-behaved $f$ and any $1 \leq n \leq N$ we have

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^n f(u_i) \right| \leq \frac{1}{n} D(n, \mathcal{C}_2) V(f), \tag{2}$$

where $V(f)$ is a measure of smoothness of $f$ — in two dimensions, $V(f)$ turns out to be the arclength

$$\int_0^1 |f'(x)|dx$$

of $f$ in $[0,1]$. The Koksma-Hlawka inequality is notable as the error in the estimation of $\int f$ is decoupled into a "variation" term, depending only on $f$, and a "discrepancy" term which is completely independent of $f$.

In this lecture, we continued along this path and outlined the analogues of the above results in $d$-dimensional star discrepancy for arbitrary fixed $d > 0$. Our ambient space is now the $d$-cube $[0,1)^d$; let $P \subseteq [0,1)^d$ be a finite collection of points in $[0,1)^d$, and let $A \subseteq [0,1)^d$ be Lebesgue-measurable. We define the discrepancy of $P$ with respect to $A$ to be

$$D(P, A) = |P| \operatorname{vol}(A) - |P \cap A|$$

which can be interpreted as the deviation of the overlap between $P$ and $A$ from a uniformly random $\mathbf{P}$ and $A$. Given a collection $\mathcal{A}$ of Lebesgue-measurable sets we define the discrepancy of $P$ to be

$$D(P, \mathcal{A}) = \sup_{A \in \mathcal{A}} |D(P, A)|,$$

[1] To be concrete, we can assume that "well-behaved" means any differentiable $f$ with a Riemann-integrable first derivative.

and we define

$$D(n,\mathcal{A}) = \inf_{P:|P|=n} D(P,\mathcal{A}).$$

Define the *d-dimensional corners* $\mathcal{C}_d := \{C_y \mid y \in [0,1)^d\}$ to be the collection of all sets

$$C_y = \{x \in [0,1)^d \mid \forall 1 \le i \le d : 0 \le x_i \le y_i\}$$

for $y \in [0,1)^d$. It turns out that the two-dimensional Koksma-Hlawka inequality (2) generalizes to the higher dimensional case.

**Theorem 1.** *For any positive integer $N$ there is a sequence of points $u_1, u_2, \ldots, u_N \in [0,1)^d$ such that for each well-behaved $f : [0,1)^d \to \mathbf{R}$ and for each positive integer $n \le N$ we have*

$$\left| \int_{[0,1)^d} f(x)dx - \frac{1}{n}\sum_{i=1}^{n} f(u_i) \right| \le \frac{1}{n} D(n, \mathcal{C}_{d+1}) V(f),$$

*where $V(f)$ is the total variation in the sense of Hardy and Krause.*

Whereas this "smoothness measure" $V(f)$ is quite simple when $f$ takes a single argument, in general it is a much more complicated function and we will not define it here. However, this is good motivation for studying higher dimensional star discrepancy; so, what upper and lower bounds are known for $D(n,\mathcal{C}_d)$?

For upper bounds, it turns out that the construction from last lecture using the van der Corput sequence can be generalized. For completeness we give the construction here. In the van der Corput sequence we consider the *bit reversal function* $g_2 : \mathbf{N} \to [0,1)$ as follows. For any natural $a$ let $a_0^{(2)} \cdots a_{k-1}^{(2)}$ be its corresponding binary expression (note that $k$ depends on $a$, which we suppress for the sake of brevity). Then

$$g_2(a) = \sum_{i=0}^{k-1} \frac{a_i^{(2)}}{2^{i+1}}$$

is obtained by "reversing" the binary expression of $a$ and placing the reversed expression after the radix point. The van der Corput sequence of length $n$ is $g_2(0), g_2(1), \ldots g_2(n-1)$, and the corresponding low-discrepancy set $P$ for $\mathcal{C}_2$ is

$$P = \{(i/n, g_2(i)) \mid 0 \le i \le n-1\}.$$

We can naturally generalize this: if $m \ge 2$ is an integer we can write $a = \sum_{i=0}^{k-1} a_i^{(m)} m^i$ in "base $m$", where $0 \le a_i^{(m)} \le m-1$ for all $i$. Then define

$$g_m(a) = \sum_{i=0}^{k-1} \frac{a_i^{(m)}}{m^{i+1}}.$$

Following Matousek [3] we define the *Halton-Hammersley sequence* of length $n$ as follows: first fix $d-1$ distinct primes[2] $p_1, p_2, \ldots, p_{d-1}$, and then the $i$th point in the sequence is defined by

$$(i/n, g_{p_1}(i), g_{p_2}(i), \ldots, g_{p_d - 1}(i)).$$

[2] In fact, any collection of pairwise coprime integers will do.

It can be shown (by a similar proof as in the two-dimensional case) that the Halton-Hammersley sequence certifies the upper bound $D(n, \mathcal{C}_d) = O(\log^{d-1} n)$.

However, for $d > 2$ we no longer have tight bounds on the star discrepancy — it is known that Roth's method generalizes to give $D(n, \mathcal{C}_d) = \Omega(\log^{(d-1)/2} n)$, but the best current lower bound is of the form $D(n, \mathcal{C}_d) = \Omega(\log^{(d-1)/2+\varepsilon_d} n)$ where $\varepsilon_d > 0$ tends to 0 as $d \to \infty$ [2]. We record these observations as a theorem.

**Theorem 2.** *For all $n \geq 0$, $d \geq 3$, we have $D(n, \mathcal{C}_d) = O(\log^{d-1} n)$ and $D(n, \mathcal{C}_d) \geq \Omega(\log^{(d-1)/2+\varepsilon_d} n)$ for some $\varepsilon_d > 0$.*

**Open Problem:** Close the gap between the upper and lower bounds on $D(n, \mathcal{C}_d)$ for $d \geq 3$.

## 2 Combinatorial Discrepancy

We now consider *combinatorial discrepancy*, which is an interesting measure in and of itself (and is also related to the usual continuous discrepancy, as we will see). Let $\mathcal{U}$ be a set and let $\mathcal{S} \subseteq 2^{\mathcal{U}}$ a family of subsets of $\mathcal{U}$; together the pair $(\mathcal{U}, \mathcal{S})$ is called a *set system*. Given a colouring $\chi : \mathcal{U} \to \{-1, 1\}$ of the elements of $\mathcal{U}$ with $\pm 1$ we define

$$\operatorname{disc}(\chi, \mathcal{S}) = \max_{S \in \mathcal{S}} |\chi(S)|$$

where $\chi(S) = \sum_{j \in S} \chi(j)$. The *discrepancy* of $\mathcal{S}$ is

$$\operatorname{disc}(\mathcal{S}) = \min_{\chi} \operatorname{disc}(\chi, \mathcal{S})$$

where the minimum is taken over all colourings $\chi : \mathcal{U} \to \{-1, 1\}$.

First we link combinatorial discrepancy with continuous discrepancy. If $\mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{S}$ is a collection of subsets of $\mathcal{U}$ then the *restriction of $\mathcal{S}$ to $\mathcal{V}$* is the collection

$$\mathcal{S}|_{\mathcal{V}} = \{S \cap \mathcal{V} \mid S \in \mathcal{S}\}.$$

**Lemma 3** (See Transference Lemma, Prop. 1.8 in [3])**.** *Let $\mathcal{A}$ be a class of Lebesgue measurable sets in $[0, 1)^d$ such that $[0, 1)^d \in \mathcal{A}$ and suppose $D(n, \mathcal{A}) = o(n)$. Additionally, suppose that*

$$\max_{\substack{P \subseteq [0,1)^d \\ |P| = n}} \operatorname{disc}(\mathcal{A}|_P) \leq f(n)$$

*holds for all $n > 0$, where $f(n)$ is a function satisfying $f(2n) \leq (2 - \delta) f(n)$ for all $n$ and some fixed $\delta > 0$. Then $D(n, \mathcal{A}) = O(f(n))$.*

Before proving the lemma, let us recall a useful auxiliary claim (proved in Lecture 1) which relates combinatorial discrepancy and continuous discrepancy. If $(\mathcal{U}, \mathcal{S})$ is a set system with $\mathcal{U}$ finite, and $U$ is a subset of $\mathcal{U}$, we say that $U$ is an *$\varepsilon$-approximation* of $(\mathcal{U}, \mathcal{S})$ if

$$\left| \frac{|U \cap S|}{|U|} - \frac{|S|}{|\mathcal{U}|} \right| \leq \varepsilon$$

for all $S \in \mathcal{S}$.

**Proposition 4** (Lemma 1.6.ii in [3]). *Let $(\mathcal{U}, \mathcal{S})$ be a set system with $|\mathcal{U}| = 2n$ and $\mathcal{U} \in \mathcal{S}$. If $\operatorname{disc}(\mathcal{S}) \leq \varepsilon n$ then there is a set $U \subseteq \mathcal{U}$ with $|U| = n$ such that $U$ is an $\varepsilon$-approximation of $(\mathcal{U}, \mathcal{S})$.*

The previous proposition will be applied in the proof of the Transference Lemma as follows: by assumption $D(n, \mathcal{A}) = o(n)$, which means that we can choose a large (but finite) set of points $P_0$ so that $D(|P_0|, \mathcal{A})/|P_0| \leq f(n)/n$. By restricting $\mathcal{A}$ to $P_0$ we can consider the finite set system $(P_0, \mathcal{A}|_{P_0})$, which by assumption will have a colouring $\chi$ with discrepancy $f(n)$. Then we can apply the previous lemma repeatedly to "sparsify" $P_0$ without damaging the discrepancy too much. After sparsifying the set roughly $\log|P_0| - \log n$ times we will end up with a set of size $n$ that is a good approximation of $D(n, \mathcal{A})$, proving the lemma.

*Proof of Lemma 3.* Set $\varepsilon = f(n)/n$, and choose $k$ large enough so that

$$\frac{D(2^k n, \mathcal{A})}{2^k n} \leq \varepsilon,$$

and note that this is possible since $D(n, \mathcal{A}) = o(n)$ by assumption. Let $P_0 \subseteq [0,1)^d$ with $|P_0| = 2^k n$ be chosen so that $D(P_0, \mathcal{A}|_{P_0}) = D(P_0, \mathcal{A}) \leq \varepsilon$, and consider the — necessarily finite — $\operatorname{disc}(\mathcal{A}|_{P_0})$.

By assumption, there is a $\pm 1$ colouring $\chi$ of $P_0$ so that $\operatorname{disc}(\chi, \mathcal{A}|_{P_0}) \leq f(2^k n)$. Applying Proposition 4 there is a set of points $P_1 \subseteq P_0$ with $|P_1| = |P_0|/2 = 2^{k-1}n$ such that $P_1$ is an $\varepsilon_0$-approximation of the set system $(P_0, \mathcal{A}|_{P_0})$ for $\varepsilon_0 = f(2^k n)/2^k n$. Applying this step $k$ more times yields $k$ sets $P_1, P_2, \ldots, P_k$ where $|P_i| = 2^{k-i}n$ and such that $P_i$ is an $\varepsilon_i$-approximation of the set system $(P_{i-1}, \mathcal{A}|_{P_{i-1}})$ for $\varepsilon_i = f(2^{k-i}n)/2^{k-i}n$. By the definition of an $\varepsilon$-approximation and the triangle inequality, it follows that $P_0$ is a $\nu$-approximation of $D(n, \mathcal{A})$, where

$$\nu = \varepsilon + \sum_{i=0}^{k} \varepsilon_i.$$

Using the assumption that $f(2n) \leq (2-\delta) f(n)$ and employing a geometric series we get

$$\begin{aligned}
\nu = \varepsilon + \sum_{i=0}^{k} \varepsilon_i &= \frac{f(n)}{n} + \sum_{i=0}^{k} \frac{f(2^{k-i}n)}{2^{k-i}n} \\
&\leq \frac{f(n)}{n} + \sum_{i=0}^{k} \frac{f(n)(2-\delta)^{k-i}}{2^{k-i}n} \\
&\leq \frac{f(n)}{n}\left(1 + \sum_{i=0}^{\infty}\left(\frac{2-\delta}{2}\right)^i\right) = O\left(\frac{f(n)}{n}\right),
\end{aligned}$$

thus justifying the somewhat odd requirement on $f$. Since $P_0$ is an $\nu$-approximation of $D(n, \mathcal{A})$ it easily follows that $D(P_0, \mathcal{A}) \leq n\nu = O(f(n))$, proving the lemma. $\square$

With the Transference Lemma in our pocket we can convert combinatorial discrepancy upper bounds into continuous discrepancy upper bounds. To this end, fix $\mathcal{U} = [n]$, let $\mathcal{S}$ be a family of subsets of $\mathcal{U}$ with $|\mathcal{S}| = m$. What kind of upper bounds can we expect for the combinatorial discrepancy of $(\mathcal{U}, \mathcal{S})$?

An instructive example to consider is a uniformly random colouring. As the next proposition shows, if $m$ is small then most colourings actually have fairly good discrepancy.

**Proposition 5.** *Let $\chi : \mathcal{U} \to \{-1, 1\}$ be a random colouring such that each $\chi(j)$ is uniformly and independently chosen over $\pm 1$. Then with probability at least $1/2$ we have $\operatorname{disc}(\chi, \mathcal{S}) = O(\sqrt{n \log m})$.*

*Proof.* This is a textbook application of the probabilistic method. For any $S \in \mathcal{S}$ we have $\mathbb{E}[\chi(S)] = 0$ and $\operatorname{Var}(\chi(S)) = \mathbb{E}[\chi(S)^2] = |S|$. Then a Chernoff bound gives us

$$\Pr[|\chi(S)| \geq t\sqrt{|S|}] \leq 2e^{-t^2/2}.$$

Setting $t = \sqrt{4 \ln m}$ and using a union bound we get

$$\Pr[\exists S \in \mathcal{S} : |\chi(S)| \geq \sqrt{4|S| \ln m}] \leq 1/2,$$

thus with probability at least $1/2$ we have that the discrepancy of $\chi$ is at most $O(\sqrt{n \log m})$. □

Is this upper bound the best we can hope for for all $m$? Often times in extremal combinatorics an appropriately chosen "random" configuration gives optimal bounds, but for combinatorial discrepancy it turns out we can eliminate the $\log m$ term when $m = O(n)$.

**Theorem 6** (Six Standard Deviations Suffice [5])**.** *Let $(\mathcal{U}, \mathcal{S})$ be any set system with $|\mathcal{U}| = n$ and $|\mathcal{S}| = m \geq n$. Then $\operatorname{disc}(\mathcal{S}) \leq O(\sqrt{n \log(2m/n)})$. In particular, if $m = n$ then $\operatorname{disc}(\mathcal{S}) \leq 6\sqrt{n}$.*

In what other cases can we achieve better upper bounds on the combinatorial discrepancy? A natural case to consider is when the set system $(\mathcal{U}, \mathcal{S})$ have bounded degree in the following sense.

**Definition 7.** *Let $(\mathcal{U}, \mathcal{S})$ be a finite set system. The* degree *of $(\mathcal{U}, \mathcal{S})$ is*

$$\Delta(\mathcal{S}) = \max_{j \in \mathcal{U}} |\{S \in \mathcal{S} \mid j \in S\}|.$$

Beck and Fiala [1] provided a much better (and algorithmically effective) upper bound for set systems with small degree.

**Theorem 8.** *Let $(\mathcal{U}, \mathcal{S})$ be a set system. Then $\operatorname{disc}(\mathcal{S}) \leq 2\Delta(\mathcal{S}) - 1$.*

*Proof.* Let $(\mathcal{U}, \mathcal{S})$ be a set system, and for brevity let $\Delta = \Delta(\mathcal{S})$. Let $n = |\mathcal{U}|$. We provide an iterative rounding procedure to construct a colouring $\chi$ satisfying $\operatorname{disc}(\chi, \mathcal{S}) \leq 2\Delta - 1$. To be more precise, we construct a sequence of colourings $\chi_0, \chi_1, \ldots, \chi_T$ so that for all $j \in \mathcal{U}$:

1. $\chi_0(j) = 0$,
2. for all $i = 0, 1, \ldots, T$, $\chi_i(j) \in [-1, +1]$,
3. $\chi_T(j) \in \{-1, +1\}$.

At each time $t = 1, 2, \ldots, T$ we specify two sets: first, the *active* indices at $t$ are

$$A_t := \{j \mid -1 < \chi_t(j) < +1\},$$

and second, the *dangerous* sets at $t$ are

$$\mathcal{D}_t = \{S \in \mathcal{S} \mid |S \cap A_t| > \Delta\}.$$

Our goal at time $t + 1$ is to construct a $\chi_{t+1}$ so that

1. $\forall S \in \mathcal{D}_t,\ \chi_{t+1}(S) = 0$,
2. $\forall j \notin A_t,\ \chi_{t+1}(j) = \chi_t(j)$,
3. $A_{t+1} \subsetneq A_t$.

First we prove the theorem, assuming that we can construct such a sequence of colourings. It is clear that $T \leq n$, since we fix at least one active variable in each iteration and fixed variables never become active in later iterations. Now, choose an arbitrary $S \in \mathcal{S}$ and suppose that $S$ becomes safe at time $t$. Then by the triangle inequality

$$|\chi_T(S)| = |\chi_T(S) + \chi_t(S) - \chi_t(S)| \leq |\chi_t(S)| + |\chi_T(S) - \chi_t(S)|.$$

Since $S$ became safe at iteration $t$ we have $\chi_t(S) = 0$. Applying this and using the definition of the dangerous sets $\mathcal{D}_t$ we have

$$|\chi_T(S)| = |\chi_T(S \cap A_t) - \chi_t(S \cap A_t)| \leq |\chi_T(S \cap A_t)| + |\chi_t(S \cap A_t)| < 2\Delta,$$

where the strict inequality follows since $-1 < \chi_t(j) < +1$ for all $j \in S \cap A_t$. But, the discrepancy must be integral, so we must have $|\chi_T(S)| \leq 2\Delta - 1$.

Next we move on to constructing the sequence of colourings. Fix any time $t \in \{0, 1, 2, \ldots, T\}$, and consider the system of equations

$$\forall S \in \mathcal{D}_t : \sum_{j \in S} x_j = 0$$

where there is one variable $x_j$ for each active $j \in A_t$. Note that

$$|\mathcal{D}_t|\Delta < |\{(j, S) \mid j \in S \cap A_t, S \in \mathcal{D}_t\}| \leq |A_t|\Delta,$$

where the upper bound follows since each element $j$ is contained in at most $\Delta$ sets, and the lower bound follows since each set $S \in \mathcal{D}_t$ is dangerous and so $|S \cap A_t| > \Delta$. This means that $|\mathcal{D}_t| < |\mathcal{A}_t|$, and thus the above system of equations has more variables than constraints. Let $y^*$ be a non-zero solution to the system: then we can define

$$\chi_{t+1}(j) = \begin{cases} \chi_t(j) & \text{if } j \notin A_t \\ \chi_t(j) + \alpha y_j^* & \text{otherwise} \end{cases}$$

where $\alpha$ is the largest real so that $\chi_{t+1}(j) \in [-1, +1]$ for all $j$. There must be at least one $j$ that becomes integral after this "update", and it follows that $A_{t+1} \subsetneq A_t$. The other two requirements follow directly from the construction. $\square$

In the same paper, Beck and Fiala made the following conjecture, which remains open today:

**Open Problem:** Prove or disprove the upper bound $\text{disc}(\mathcal{S}) \leq O(\sqrt{\Delta(\mathcal{S})})$.

# References


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