Rounding and Approximation Algorithms

CSC 473 Advanced Algorithms

Computer Science
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Approximation Algorithms

- Many natural and important optimization problems are NP-Hard
  - Vertex/Set cover, Max SAT, Max Cut, Sparsest Cut, Traveling Salesman, …
- No worst-case polynomial time exact algorithms
- Ways around the hardness:
  - Approximation algorithms: output an approximately optimal solution in worst-case polynomial time
  - Algorithms that are efficient on special instances: e.g. Max Cut in planar graphs
  - Algorithms that are exponential in some parameter: $2^{O(k)} poly(n)$ time algorithm to find a path of length k.

Vertex Cover

- **Min Vertex Cover (VC):** on input a graph $G = (V, E)$ and vertex weights $w \in \mathbb{R}^V$ find set $C \subseteq V$ such that
  - for all $(u, v) \in E$, $u \in C$, or $v \in C$, or both
  - $w(C) = \sum_{u \in C} w_v$ is minimized

- NP-hard for non-bipartite graphs
- Factor 2 approximation
  - Formulate as an IP, and relax to an LP
  - Round (possibly fractional) optimal LP solution to a $\{0,1\}$ solution

LP Relaxation

- Let $OPT(G,w) = \min\{w(C): C \text{ is a vertex cover}\}$

$$\min \sum_{u \in V} w_u x_u$$
s.t. $\quad x_u + x_v \geq 1 \quad \forall (u,v) \in E$
$\quad x_u \in \{0,1\} \quad \forall u \in V$

$$\min \sum_{u \in V} w_u y_u$$
s.t. $\quad y_u + y_v \geq 1 \quad \forall (u,v) \in E$
$\quad 0 \leq y_u \leq 1 \quad \forall u \in V$

- $LP(G,w) =$ value of the LP
- $OPT(G,w) =$ value of the IP $\geq LP(G,w)$

Deterministic Rounding

- Will show: $LP(G,w) \leq OPT(G,w) \leq 2 \cdot LP(G,w)$
- Let y be an optimal LP solution and define

$$C = \{u \in V : y_u \geq 1/2\}$$

$$\min \sum_{u \in V} w_u y_u$$

s.t. $\quad y_u + y_v \geq 1 \quad \forall (u,v) \in E$

$$0 \leq y_u \leq 1 \quad \forall u \in V$$

- C is a vertex cover
  - If $(u,v) \in E$ then $\max\{y_u, y_v\} \geq \frac{1}{2}$
- C is a 2-approximate solution: Let $x_u = 1 \Leftrightarrow u \in C$. Then $x_u \leq 2y_u$ and

$$w(C) = \sum_{u \in V} w_u\, x_u \leq 2 \sum_{u \in V} w_u\, y_u = 2 \cdot LP(G,w) \leq 2 \cdot OPT(G,w)$$

Set Cover

- **Min Set Cover:** on input $S_1, S_2, \dots, S_m \subseteq [n]$ where $\bigcup_{i=1}^{m} S_i = [n]$, and weights $w \in \mathbb{R}^m$, find set $C \subseteq [m]$ such that
  - $\bigcup_{i \in C} S_i = [n]$
  - $w(C) = \sum_{i \in C} w_i$ is minimized

- Vertex cover is a special case
- Factor $O(\log n)$ approximation
  - No better factor possible, unless P=NP

LP Relaxation

- Let $OPT = \min\{w(C): C \text{ is a vertex cover}\}$

Every element $j \in [n]$ is covered

$$\min \sum_{i=1}^{m} w_i x_i$$
$$\text{s.t.} \quad \sum_{i: j \in S_i} x_i \geq 1 \quad \forall j \in [n]$$
$$x_i \in \{0,1\} \quad \forall i \in [m]$$

$$\min \sum_{i=1}^{m} w_i y_i$$
$$\text{s.t.} \quad \sum_{i: j \in S_i} y_i \geq 1 \quad \forall j \in [n]$$
$$0 \leq y_i \leq 1 \quad \forall i \in [m]$$

- LP = value of the LP
- OPT = value of the IP $\geq LP$

Randomized Rounding

$$\min \sum_{i=1}^{m} w_i y_i$$

s.t. $\sum_{i:j\in S_i} y_i \geq 1 \quad \forall j \in [n]$

$0 \leq y_i \leq 1 \quad \forall i \in [m]$

- $y =$ optimal LP solution
- For $t = 1, \ldots, \ell = \ln(2n)$
  - $C_t = \emptyset$
  - For $i = 1, \ldots, m$: Add i to C_t with probability y_i
- $C = C_1 \cup \cdots \cup C_\ell$
- **Claim 1**. $\mathbb{E}[w(C)] \leq \ell \cdot LP$
  - $Z_{t,i} = 1$ if $i \in C_t$, 0 otherwise. $\mathbb{E}\left[Z_{t,i}\right] = \mathbb{P}(Z_{t,i} = 1) = y_i$.

$$\mathbb{E}[w(C)] \leq \mathbb{E}[w(C_1)] + \cdots + \mathbb{E}[w(C_\ell)] = \sum_{t=1}^{\ell} \sum_{i=1}^{m} w_i \mathbb{E}\left[Z_{t,i}\right] = \ell \sum_{i=1}^{m} w_i y_i$$

Randomized Rounding

$$\min \sum_{i=1}^{m} w_i y_i$$
$$\text{s.t.} \quad \textstyle\sum_{i:j\in S_i} y_i \geq 1 \quad \forall j \in [n]$$
$$0 \leq y_i \leq 1 \quad \forall i \in [m]$$

- $y =$ optimal LP solution
- For $t = 1, \ldots, \ell = \ln(2n)$
  - $C_t = \emptyset$
  - For $i = 1, \ldots, m$: Add i to C_t with probability y_i
- $C = C_1 \cup \cdots \cup C_\ell$
- **Claim 2**. $\mathbb{P}(C \text{ is a set cover}) \geq \frac{1}{2}$

$$\forall j:\ \mathbb{P}\left(\forall t: j \notin \bigcup_{i\in C_t} S_i\right) = \prod_{t=1}^{\ell} \prod_{i:j\in S_i} (1 - y_i) \leq e^{-\ell \sum_{i:j\in S_i} y_i} \leq e^{-\ell} = \frac{1}{2n}$$

  - By a union bound, the probability that some j is not covered is $\leq n \cdot \frac{1}{2n} = \frac{1}{2}$

Randomized Rounding

What if $w(C)$ is small only when C is not a cover?

- **Claim 1**. $\mathbb{E}[w(C)] \leq \ell \cdot LP$
- **Claim 2**. $\mathbb{P}(C \text{ is a set cover}) \geq \frac{1}{2}$

- Repeat until C is a set cover. Expected weight is $\mathbb{E}[w(C)|C \text{ a cover}]$

$$\mathbb{E}[w(C)] = \mathbb{E}[w(C)|C \text{ a cover}]\mathbb{P}(C \text{ a cover})$$
$$+\mathbb{E}[w(C)|C \text{ not a cover}]\mathbb{P}(C \text{ not a cover})$$
$$\geq \mathbb{E}[w(C)|C \text{ a cover}]/2$$

$$\mathbb{E}[w(C)|C \text{ a cover}] \leq 2\mathbb{E}[w(C)] \leq 2\ell \cdot LP = O(\log n) \cdot LP$$